SUB-RIEMANNIAN MANIFOLDS ARE NOT CD (with L. Rizzi)

What is a sub-Riemannian manifold?

DISTRIBUTION

Fix a smooth manifold M, with $\dim M = n \geq 2$.
Fix $\mathcal{F} = \{X_1, X_2, \dots, X_L\}$, $L \geq 1$, smooth vector fields on M.
The sub-R distribution induced by $\mathcal{F}$ is

$$\mathcal{D} = \bigsqcup_{x \in M} \mathcal{D}_x \quad , \quad \mathcal{D}_x = \operatorname{span}\{X_1|_x, \dots, X_L|_x\} \subset T_x M \quad , x \in M.$$

Note that $x \mapsto \dim \mathcal{D}_x$ may vary!

Assumption (Hörmander / bracket-generating) We assume that

$$T_x M = \{ X|_x \ : \ X \in \underbrace{\operatorname{Lie}(\mathcal{F})}_{\text{Lie alg. gen. by } \mathcal{F}} \} \quad \forall x \in M \qquad (H)$$

METRIC STRUCTURE

For $x \in M$ and $V \in \mathcal{D}$, we set

$$\|V\|_x = \min \left\{ |u| \ : \ u \in \mathbb{R}^L \ \text{s.t.} \ V|_x = \sum_{i=1}^{L} u_i X_i|_x \ , \ X_i \in \mathcal{F} \right\}$$

Note that $x \mapsto \|\cdot\|_x$ satisfies the parall. law

$\Longrightarrow$ it is induced by $g_x : \mathcal{D}_x \times \mathcal{D}_x \longrightarrow \mathbb{R}$ $\quad \forall x \in M$.

We define the CC distance of $x, y \in M$ as

$$d_{CC}(x,y) = \inf \left\{ \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)} \, dt \ : \ \begin{array}{l} \gamma \in \operatorname{Lip}_{loc} \ \text{(in charts)} \\ \dot\gamma(t) = \sum_{i=1}^{L} u_i(t) X_i|_{\gamma(t)} \\ \quad \text{for a.e. } t \in [0,1] \\ \text{with } u \in L^\infty([0,1]; \mathbb{R}^L) \\ \gamma(0) = x, \ \gamma(1) = y \end{array} \right\}$$

Theorem (Chow–Rashevskii) $(H) \Longrightarrow d_{CC}$ is a distance!

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DIFFERENTIAL OPERATORS

Set $\Gamma(\mathcal{D}) = \mathcal{C}^\infty$-module gen. by $\mathcal{F}$.

Fix $u \in \mathcal{C}^\infty(M)$. We define the (horizontal) gradient $\nabla u \in \Gamma(\mathcal{D})$ as

$$g(\nabla u, V) = du(V) \quad \forall\, V \in \Gamma(\mathcal{D}).$$

Fact: $\nabla u = \sum_{i=1}^{L} (X_i u) X_i$ with $\|\nabla u\|^2 = \sum_{i=1}^{L} (X_i u)^2$

(even if $\mathcal{F}$ is not lin. indep.) #

We equip M with a positive smooth measure m.

We define the sub-laplacian $\Delta u \in \mathcal{C}^\infty$ as

$$\int_M g(\nabla u, \nabla v)\, dm = -\int_M v \Delta u \, dm \quad \forall\, v \in \mathcal{C}^\infty_c(M)$$

Fact: $\Delta u = \sum_{i=1}^{L} \left(X_i^2 u + (X_i u)\, \mathrm{div}_m(X_i) \right)$

(even if $\mathcal{F}$ is not lin. indep.) $\int_M v \,\mathrm{div}_m(V)\, dm = -\int_M g(\nabla v, V)\, dm$ #

FAILURE OF BE/CD

Definition (Bakry–Émery inequality)

We say that (M, d_{cc}, m) satisfies $BE(K,\infty)$, $K \in \mathbb{R}$, if

$$\frac{1}{2} \Delta\left(\|\nabla u\|^2\right) \geq g(\nabla u, \nabla \Delta u) + K \|\nabla u\|^2 \quad \forall\, u \in \mathcal{C}^\infty(M).$$

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Main Theorem [Rizzi – S., 2023]

(M, d_{cc}, m) satisfies $BE(K,\infty)$ $\Longrightarrow$ $\mathrm{rank}\, \mathcal{D}_x = \dim M \quad \forall\, x \in M$

(i.e. (M,g) is Riemannian!) #

Fact: (M,d,m) inf. Hilbertian $\oplus$ $CD(K,\infty)$ $\overset{\text{[Gigli]}}{\Longrightarrow}$ $BE(K,\infty)$ $\not\Leftarrow$

[Sturm & Lott–Villani]
2006 2009

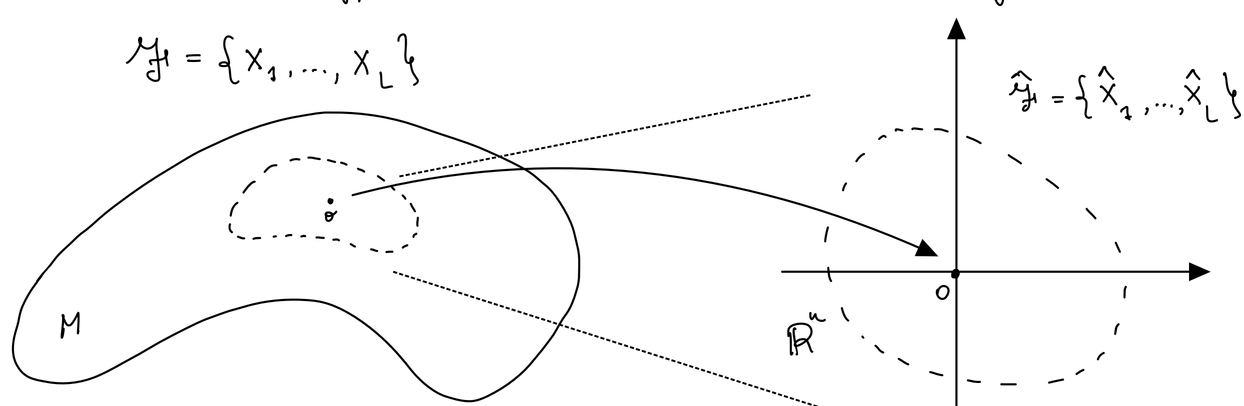
Fact: (M, d_{cc}, m) is inf. Hilbertian (even with m just Radon).
[Le Donne – Lučić – Pasqualetto] 2022 #

Our result implies several know partial results:

- [Driver – Melcher, 2005] for $\mathbb{H}^1$ (Heisenberg group)
- [Juillet, 2009] for $\mathbb{H}^n$
- [Juillet, 2010] for Grushin plane
- [Ambrosio – S., 2020] for Carnot groups $\rightarrow$ reverse Poincaré
- [Huang – Sun, 2020] for equireg. distr. $\rightarrow$ embeddings
- [Juillet, 2021] for rank $\mathcal{D} \lneq \dim M \rightarrow$ BM
- [Magnabosco – Rossi, 2023] for almost-R, $\dim M = 2$ or strongly reg.
  ↳ via Cavalletti – Mondino decomposition

PROOF OF MAIN THEOREM (SKETCH)

Pass to nilpotent approximation aka Gromov's metric tangent cone:



$$(M, \mathcal{F}, d_{cc}, m, o) \longrightarrow (\mathbb{R}^n, \hat{\mathcal{F}}, d_{cc}, \mathcal{L}^n, 0)$$

(homogeneous structure)

Facts: (1) the generated Lie algebra is stratifies, as

$$\underline{g} = \mathrm{Lie}(\hat{\mathcal{F}}) = \underline{g}^1 \oplus \underline{g}^2 \oplus \dots \oplus \underline{g}^s \quad \text{for some } s \in \mathbb{N}$$

$$\text{with } \underline{g}^{i+1} = [\underline{g}^1, \underline{g}^i] \quad \forall i = 1, \dots, s-1.$$

(2) also $\underline{h} = \{ \hat{X} \in \underline{g} : \hat{X}|_0 = 0 \} = \underline{h}^1 \oplus \dots \oplus \underline{h}^s$.

③ $(\mathbb{R}^n, \hat{\mathcal{F}}) \sim G/H$, $G = \exp \underline{g}$, $H = \exp \underline{h}$. #

Fact: $\hat{\Delta} u = \sum_{i=1}^{L} \hat{X}_i^2 u \quad \forall u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Blow-upping $BE(K,\infty)$, we get $BE(0,\infty)$ on $(\mathbb{R}^n, \hat{\mathcal{F}})$:

$$(*) \quad \sum_{i=1}^{L} \hat{X}_i u \left(\hat{X}_{ijj} u - \hat{X}_{jji} u \right) - (\hat{X}_{ij} u)^2 \leq 0 \quad \forall u \in \mathcal{C}^\infty(\mathbb{R}^n).$$

where $X_{ijk} = X_i X_j X_k$.

Fact: X_I kills hom. polynomials of degree $|I|$ #

Take $u = \alpha + \gamma$ with $\deg \alpha = 1$ and $\deg \gamma \geq 3$ in $(*)$ to get

$$\sum_{i,j=1}^{L} \hat{X}_i \alpha \left(\hat{X}_{ijj} \gamma - \hat{X}_{jji} \gamma \right) = \sum_{i=1}^{L} \hat{X}_i \alpha \, [\hat{X}_i, \hat{\Delta}](\gamma) \leq 0$$

(since $X_{|I|} \alpha = 0 \quad \forall |I| \geq 2$). Hence (using $-\alpha$ in place of α)

$$(**) \quad \sum_{i=1}^{L} \underset{\substack{\uparrow \\ \text{constant}}}{\hat{X}_i \alpha} \cdot [\hat{X}_i, \hat{\Delta}](\gamma) = \left[\sum_{i=1}^{L} \hat{X}_i \alpha X_i, \hat{\Delta} \right](\gamma) = 0$$

Now consider the map $\phi : \mathbb{P} \longrightarrow \mathcal{I}(\mathcal{D})$ as

$$\phi[\alpha] = \hat{\nabla} \alpha = \sum_{i=1}^{L} (\hat{X}_i \alpha) X_i \qquad \alpha \in \mathbb{P} = \text{polynomial with lowest weight}$$

As a consequence, any $\hat{X} \in \underline{i} = \phi[\mathbb{P}]$ satisfies $[\hat{X}, \hat{\Delta}] = 0$.

We now study $\underline{i} \subset \underline{g}^1$. We prove that

(a) $\underline{g}^1 = \underline{i} \oplus \underline{h}^1$; $\longrightarrow$ algebraic proof

(b) each $\hat{X} \in \underline{i}$ is Killing. $\longrightarrow$ exploit Hamiltonian #

From (a) and (b) one can infer that

$$[\underline{i},\underline{i}] \subset \underline{h}^2 \quad \text{and} \quad [\underline{i},\underline{h}^1] \subset \underline{h}^2 .$$

Hence, by induction, $[\underline{i},\underline{h}^j] \subset \underline{h}^{j+1}$, and so

$$\underline{g} = \underline{g}^1 \oplus \underline{h}^2 \oplus \dots \oplus \underline{h}^s .$$

But then $\underline{g}|_0 = \underline{g}^1|_0$, so $\underline{g}$ must be commutative!

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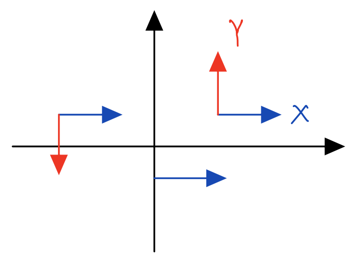
Remark The assumption that m is positive cannot be dropped! Consider $M = \mathbb{R}^2$ with $\mathcal{F} = \{X, Y\}$,

$$X = \partial_x \quad , \quad Y = x\,\partial_y$$



with metric (for $x \neq 0$)

$$g = dx \otimes dx + \frac{1}{x^2}\, dy \otimes dy$$

and measure

$$m_p = |x|^p \, dx dy \qquad \left(p = -1 \Rightarrow m_{-1} = vol_g \right)$$

Theorem [Rizzi–S., 2023]

(i) $p \geq 0 \implies$ no $CD(K,\infty)$ $\forall K \in \mathbb{R}$ in $\mathbb{R}^2$

(ii) $p \geq 1 \implies$ yes $BE(0,\infty)$ a.e. in $\mathbb{R}^2$

(iii) $\{x \geq 0\}$ satisfies $(R)CD(0,N)$ iff

$$N \geq N_p = \frac{(p+1)^2}{p-1} + 2 \qquad (N_1 = \infty)$$

(hence for $N \geq 10$ we get no gluing in $RCD(0,N)$).

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SIMPLE EXAMPLES

Heisenberg group $\mathbb{H}^1$

In $\mathbb{R}^3$ consider $\mathcal{H} = \{X, Y, Z\}$ with

$$X = \partial_x - \frac{1}{2} y \partial_z , \quad Y = \partial_y + \frac{1}{2} x \partial_z , \quad Z = [X, Y] = \partial_z$$

for $(x, y, z) \in \mathbb{R}^3$. We have the group law

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right)$$

for $(x, y, z), (x', y', z') \in \mathbb{R}^3$.

The volume (Haar) measure is $\mu = \mathcal{L}^3 = dx\, dy\, dz$.

The structure is homog. with

$$dil_\lambda (x, y, z) = (\lambda x, \lambda y, \lambda^2 z) \qquad \forall \lambda \geq 0$$

and homog. dim. $Q = 4$.

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Grushin plane

In $\mathbb{R}^2$ consider $\mathcal{H} = \{X, Y\}$ with

$$X = \partial_x \quad \text{and} \quad Y = x \partial_y , \quad (x, y) \in \mathbb{R}^2 .$$

Degeneration at $x = 0$, with metric

$$g = dx \otimes dx + \frac{1}{x^2} dy \otimes dy , \quad x \neq 0 .$$

The volume measure is $\mu = \mathcal{L}^2 = dx\, dy$ and now

$$dil_\lambda (x, y) = (\lambda x, \lambda^2 y) \ \forall \lambda \geq 0 , \quad \text{with } Q = 3 .$$

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