On the monotonicity of perimeter of convex bodies

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Monotonicity of perimeter

Let $n \geq 2$ and $A, B \subset \mathbb{R}^n$ two *convex bodies* (compact, convex, interior $\neq \varnothing$).

$$A \subset B \implies P(A) \leq P(B)$$

Well-known inequality, dates back to the ancient Greek (Archimedes postulated it in *On the sphere and the cylinder*).

Several possible proofs:

- *Cauchy formula* for the area surface of convex bodies
- *mixed volumes* are monotone
- *projection* on convex closed sets is Lipschitz
- perimeter decreases under *intersection with half-spaces*

Problem

Lower bound on $\delta(B, A) = P(B) - P(A)$ *w.r.t. Hausdorff distance?*

Lower bound for $n = 2$

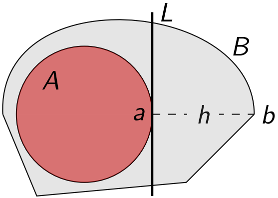
First result: La Civita-Leonetti, 2008 (non-optimal bound)

Theorem (Carozza-Giannetti-Leonetti-Passarelli di Napoli, 2015)

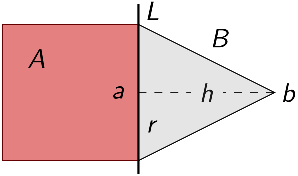
Let $A, B \subset \mathbb{R}^2$ be two convex bodies. Then

$$P(A) + \frac{2h(A,B)^2}{\sqrt{r^2 + h(A,B)^2} + r} \leq P(B),$$

where $L = \{x \in \mathbb{R}^2 : \langle b - a, x - a \rangle = 0\}$, with $a \in A$ and $b \in B$ such that $|a - b| = h(A,B)$ and $r = \frac{\mathcal{H}^1(B \cap L)}{2}$.



(a) Setting of Theorem



(b) Optimal configuration

Lower bound for $n = 3$

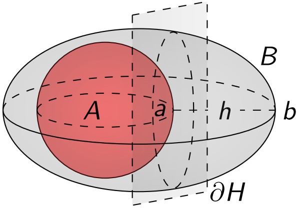
Theorem (Carozza-Giannetti-Leonetti-Passarelli di Napoli, 2016)

Let $A, B \subset \mathbb{R}^3$ be two convex bodies. Then

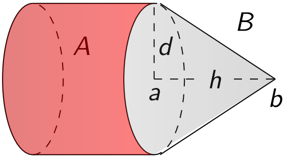
$$P(A) + \frac{\pi d h(A,B)^2}{\sqrt{d^2 + h(A,B)^2} + d} \leq P(B),$$

where $H = \{x \in \mathbb{R}^3 : \langle b - a, x - a \rangle \leq 0\}$, with $a \in A$ and $b \in B$ such that $|a - b| = h(A,B)$ and $d = \mathrm{dist}(a, \partial B \cap \partial H)$.

Note: distance $d = \mathrm{dist}(a, \partial B \cap \partial H)$ replaces the *bigger* radius $r = \sqrt{\frac{\mathcal{H}^2(B \cap \partial H)}{\pi}}$.



(a) Setting of Theorem

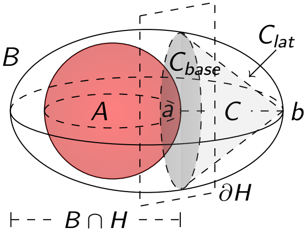


(b) Optimal configuration

Idea of the proof

Since $A \subset B \cap H$ and $C \subset B \cap \overline{H^c}$, we have

$$\begin{aligned}\delta(B,A) &= \delta(B, B\cap H) + \delta(B\cap H, A) \geq \delta(B, B\cap H)\\ &= \mathcal{H}^{n-1}(\partial B \cap H^c) - \mathcal{H}^{n-1}(\partial B \cap \partial H)\\ &\geq \mathcal{H}^{n-1}(C_{lat}) - \mathcal{H}^{n-1}(C_{base})\end{aligned}$$



The problem reduces to

$$\min\left\{\mathcal{H}^{n-1}(C_{lat}) : \text{given } h \text{ and } \mathcal{H}^{n-1}(C_{base})\right\} \qquad (\star)$$

Solution:

- $n = 2$: elementary calculus $\implies$ isosceles triangle
- $n = 3$: parametrization of C_{lat} $\implies$ right circular cone ($d > 0$ is necessary)

Problem (Leonetti's *dinner problem*)

Parametrization not easy for $n \geq 4$. Other approach to solve $(\star)$ for $n \geq 3$?

Different approach

We need to solve

$$\min\left\{\mathcal{H}^{n-1}(C_{lat}) : \text{given } h \text{ and } \mathcal{H}^{n-1}(C_{base})\right\} \qquad (\star)$$

Schwartz symmetrization

Given $E \subset \mathbb{R}^n$ convex body, define

$$E^{Sch} := \left\{ x = (x', t) \in \mathbb{R}^n : |x'| \leq \left(\frac{\mathcal{H}^{n-1}(E_t)}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \right\}.$$

Then $P(E^{Sch}) \leq P(E)$.

Case $E = C$ **cone**:

$$\text{slice} \perp \text{to } h \implies \begin{array}{c} \mathcal{H}^{n-1}((C^{Sch})_{base}) = \mathcal{H}^{n-1}(C_{base}) \\ \mathcal{H}^{n-1}((C^{Sch})_{lat}) \leq \mathcal{H}^{n-1}(C_{lat}) \end{array}$$

Result

Solution to $(\star)$ for $n \geq 3$ is a **right circular cone** with

$$\mathcal{H}^{n-1}(C_{base}) = \omega_{n-1} r^{n-1} \qquad r = \left(\frac{\mathcal{H}^{n-1}(\partial B \cap \partial H)}{\omega_{n-1}} \right)^{\frac{1}{n-1}}$$

$$\mathcal{H}^{n-1}(C_{lat}) = \omega_{n-1} r^{n-2} \sqrt{h^2 + r^2}$$

Theorem

Let $n \geq 2$. If $A \subset B$ are two convex bodies in $\mathbb{R}^n$, then

$$\mathcal{H}^{n-1}(\partial A) + \frac{\omega_{n-1} r^{n-2} h^2}{\sqrt{h^2 + r^2} + r} \leq \mathcal{H}^{n-1}(\partial B),$$

where $h = h(A, B)$ is the Hausdorff distance of A and B and

$$r = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(B \cap \partial H)}{\omega_{n-1}}}, \qquad H = \{x \in \mathbb{R}^n : \langle b - a, x - a \rangle \leq 0\},$$

with $a \in A$ and $b \in B$ such that $|a - b| = h(A, B)$.

Optimal configuration: right circular cone attached to a circular cylinder.

Generalization: Wulff perimeter

Let $n \geq 2$ and let $\Phi \colon \mathbb{R}^n \to [0,\infty)$ be a positively 1-homogeneous convex function.

Wulff Φ-perimeter

$$E \subset \mathbb{R}^n \text{ convex body}, \ \nu_E \ \textit{inner} \text{ normal} \implies P_\Phi(E) = \int_{\partial E} \Phi(\nu_E)\, d\mathcal{H}^{n-1}$$

Let $A, B \subset \mathbb{R}^n$ be two convex bodies:

$$A \subset B \implies P_\Phi(A) \leq P_\Phi(B)$$

Several possible proofs:

- *Cauchy formula* for the Wulff area surface of convex bodies
- *mixed volumes* are monotone
- Wulff perimeter decreases under *intersection with half-spaces*

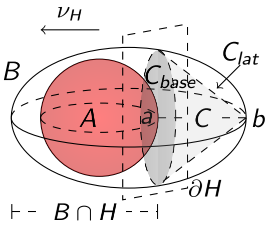
Problem

Lower bound on $\delta_\Phi(B, A) = P_\Phi(B) - P_\Phi(A)$ w.r.t. Hausdorff distance?

Same strategy

Since $A \subset B \cap H$ and $C \subset B \cap \overline{H^c}$, we have

$$\delta_\Phi(B, A) = \delta_\Phi(B, B \cap H) + \delta_\Phi(B \cap H, A) \geq \delta_\Phi(B, B \cap H)$$
$$= \int_{\partial B \cap H^c} \Phi(\nu_B)\, d\mathcal{H}^{n-1} - \Phi(\nu_H)\, \mathcal{H}^{n-1}(\partial B \cap \partial H)$$
$$\geq \int_{C_{lat}} \Phi(\nu_C)\, d\mathcal{H}^{n-1} - \Phi(\nu_H)\, \mathcal{H}^{n-1}(C_{base})$$



The problem now reduces to

$$\min\left\{ \int_{C_{lat}} \Phi(\nu_C)\, d\mathcal{H}^{n-1} : \text{given } h,\ \nu_H,\ \text{and } \mathcal{H}^{n-1}(C_{base}) \right\} \qquad (\star\star)$$

For P_Φ translations and dilations are OK, rotations are **not** OK $\implies$ ν_H matters!

Obstacle: no Schwartz symmetrization for P_Φ for general Φ!

Admissible Φ

Let $n \geq 2$ and let $\Phi \colon \mathbb{R}^n \to [0,\infty)$ be a positively 1-homogeneous convex function.

Definition

We say that Φ is *admissible* if, for each $\nu \in \mathbb{S}^{n-1}$, there exist

$$g_\nu \colon [0,\infty)^2 \to [0,\infty), \qquad \phi_\nu \colon \nu^\perp \to [0,\infty), \quad \nu^\perp = \mathrm{span}\{\nu\}^\perp$$

such that

- $g_\nu \not\equiv 0$ pos. 1-homog. and convex, $s \mapsto g_\nu(s,t)$ non-decreasing for each t;
- ϕ_ν pos. 1-homog. and convex, *coercive* on $\nu^\perp$ ($\phi_\nu(z) > 0$ $\forall z \in \nu^\perp$, $z \neq 0$);
- it holds $\Phi(x) \geq g_\nu\big(\phi_\nu(x - (x\cdot\nu)\nu), x\cdot\nu\big)$ for $x \in \mathbb{R}^n$ with $x \cdot \nu \geq 0$.

Example 1: Φ coercive on $\mathbb{R}^n$ ($\Phi_\nu(x) > 0$, $\forall x \neq 0$) $\implies$ Φ admissible, with

$$\phi_\nu(z) = |z|, \qquad g_\nu(s,t) = c\sqrt{s^2+t^2}, \quad c = \min_{|x|=1} \Phi(x) \qquad \forall \nu \in \mathbb{S}^{n-1}$$

Example 2: $\Phi(x) = |x|_p := \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$ is coercive, but for $\nu = \mathrm{e}_n$ is better

$$\phi_\nu(z) = |z|_p, \qquad g_\nu(s,t) = (s^p + t^p)^{\frac{1}{p}}$$

Admissibility $\implies$ Schwartz (from below)

For $\Phi_\nu(x) = g_\nu(\phi_\nu(x'), t)$ where $x' = x - (x \cdot \nu)\nu \in \nu^\perp$, $t = x \cdot \nu \in \mathbb{R}$, we have

Theorem (Van Schaftingen, 2006 & Baer, 2014)

Given $E \subset \mathbb{R}^n$ convex body, define

$$E^{Sch,\nu} := \left\{ x = (x', t) \in \mathbb{R}^n : x' \in \left(\frac{\mathcal{H}^{n-1}(E_t)}{\mathcal{H}^{n-1}(W_{\phi_\nu})} \right)^{\frac{1}{n-1}} W_{\phi_\nu} \right\}.$$

where $W_{\phi_\nu} \subset \nu^\perp$ is the ϕ_ν-Wulff shape in $\nu^\perp$. Then $P_{\Phi_\nu}(E^{Sch,\nu}) \le P_{\Phi_\nu}(E)$.

$\implies$ we could use this general Schwartz symm. to estimate $(\star\star)$ **from below**:

$$\int_{C_{lat}} \Phi(\nu_C)\, d\mathcal{H}^{n-1} \overset{\Phi admiss.}{\geq} \int_{C_{lat}} \Phi_{\nu_H}(\nu_C)\, d\mathcal{H}^{n-1} \overset{(\heartsuit)}{\geq} \mathcal{H}^{n-1}(W_{\phi_{\nu_H}}) r^{n-2} g_{\nu_H}(h, r)$$

where $r = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(B \cap \partial H)}{\mathcal{H}^{n-1}(W_{\nu_H})}}$ and $(\heartsuit)$ = Schwartz + *coarea*, *subdifferential*, ...

General result

We get

$$\begin{aligned}\delta_\Phi(B,A) &\geq \int_{C_{lat}} \Phi(\nu_C)\, d\mathcal{H}^{n-1} - \Phi(\nu_H)\,\mathcal{H}^{n-1}(C_{base}) \\ &\geq \mathcal{H}^{n-1}(W_{\phi_{\nu_H}}) r^{n-2} g_{\nu_H}(h,r) - \Phi(\nu_H)\,\mathcal{H}^{n-1}(W_{\phi_{\nu_H}}) r^{n-1}\end{aligned}$$

Theorem

Let $n \geq 2$ and $\Phi \colon \mathbb{R}^n \to [0,\infty)$ pos. 1-homog., convex, admissible. If $A \subset B$ are two convex bodies in $\mathbb{R}^n$, then

$$P_\Phi(A) + \mathcal{H}^{n-1}(W_{\nu_H}) r^{n-2} \big(g_{\nu_H}(h,r) - \Phi(\nu_H) r\big)^+ \leq P_\Phi(B),$$

where $h = h(A,B)$ is the Hausdorff distance of A and B and

$$r = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(B \cap \partial H)}{\mathcal{H}^{n-1}(W_{\nu_H})}}, \quad H = \{x \in \mathbb{R}^n : \langle b-a, x-a\rangle \leq 0\}, \quad \nu_H = \frac{a-b}{|a-b|},$$

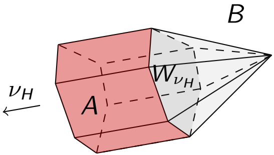
with $a \in A$ and $b \in B$ such that $|a-b| = h(A,B)$.

Remarks

Remark 1. In general $\Phi > \Phi_{\nu_H}$, so the result is **not** optimal!

When $\Phi = \Phi_{\nu_H}$, the optimal configuration is

right cone with base W_{ν_H} + cylinder with base W_{ν_H}



Remark 2. Too much technology for cones! (Baer's setting: finite perimeter sets)

Can we find a *simpler* approach? (avoid Schwartz, coarea, subdifferential...)

YES, we can follow the following scheme:

- Assume C_{base} is a **polytope** and reduce the problem to $\nu_H^\perp$.
- Apply Wulff inequality in $\nu_H^\perp$ to prove

$$\int_{C_{lat}} \Phi(\nu_C)\, d\mathcal{H}^{n-1} \geq \mathcal{H}^{n-1}(W_{\phi_{\nu_H}}) r^{n-2} g_{\nu_H}(h, r)$$

- Extend to any C_{base} by approximation.

Thank you for your attention!