Improved Lipschitz Approximation of *H*-perimeter minimizing boundaries

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Main problem

A set $E \subset \mathbb{R}^n$ is a *local perimeter minimizer* in an open set Ω if

$$P(E; B_r(x)) \le P(F; B_r(x))$$

whenever $E \triangle F \Subset B_r(x) \subset \Omega$.

Regularity theorem (De Giorgi, 1961)

E is a local perimeter minimizer in $\Omega \implies \Omega \cap \partial E$ is smooth

We now replace the group $\mathbb{R}^n$ with the **Heisenberg group** $\mathbb{H}^n$ and the Euclidean perimeter P with the (naturally induced) **horizontal perimeter** P_H.

Problem: E is a local H-perimeter minimizer in $\Omega \implies$ is $\Omega \cap \partial E$ regular?

The first step in the proof of De Giorgi's result in $\mathbb{R}^n$ is the so-called

Lip-Approximation: ∂E 'almost flat' $\implies \partial E \approx \mathrm{gr}(f)$ for some 'good' $f \in \mathrm{Lip}$

We want to establish a similar approximation result in $\mathbb{H}^n$.

Heisenberg group

The **Heisenberg group** $(\mathbb{H}^n, *)$, $n \geq 1$, is the set $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ endowed with the group law

$$(z,t) * (w,s) = (z+w, t+s+B(z,w)) \quad \forall (z,t),(w,s) \in \mathbb{H}^n,$$

where $B(z,w) = 2\,\mathrm{Im}\left(\sum_{j=1}^{n} z_j \bar{w}_j\right)$ is antisymmetric.

The identity element is $(0,0)$ and $p^{-1} = -p$ for all $p \in \mathbb{H}^n$.

The **Lie algebra** in $\mathbb{H}^n$ is given by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad j = 1, \dots, n.$$

X_j, Y_j span the **horizontal distribution** $H \subset T\mathbb{H}^n$ and $[X_j, Y_j] = -4T$.

The **box norm** of $p = (z,t) \in \mathbb{H}^n$ is $\|p\|_\infty = \max\left\{|z|, |t|^{1/2}\right\}$.

Using the induced distance d_∞, we define the **balls** $B_r(p)$ and a **spherical Hausdorff measure** $\mathcal{S}^s$, $s \geq 0$.

The induced topology is the standard one and $\mathcal{S}^{2n+2} = c_n \mathcal{L}^{2n+1}$.

H-perimeter and minimizers

The H-**perimeter** of a $\mathcal{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ in an open set $\Omega \subset \mathbb{H}^n$ is

$$P_H(E;\Omega) = \sup\left\{ \int_E \operatorname{div}_H V \; d\mathcal{L}^{2n+1} : \; V \in C^1_c(\Omega;\mathbb{R}^{2n}), \; \|V\|_g \le 1 \right\}$$

where $\operatorname{div}_H V = \sum_{j=1}^n X_j V_j + Y_j V_{j+n}$ and g is the Riemannian metric on $T\mathbb{H}^n$.

E is of **locally finite** H-**perimeter** in Ω if $P_H(E;\Omega') < \infty$ for every $\Omega' \subset\subset \Omega$.

One can define the (**reduced**) **boundary** ∂E and the H-**normal** $\nu_E \in H$, and the perimeter measure of E is given by $\mu_E = \delta(n)\, \mathcal{S}^{2n+1}|_{\partial E}$. Precisely:

Structure Theorem (Franchi, Serapioni & Serra Cassano, 2001)

E is of locally finite H-perimeter $\implies$ reduce boundary ∂E is H-rectifiable

But ∂E is **not** *Euclidean* rectifiable! (Kirchheim & Serra Cassano, 2004)

In addition, E is a **local** H-**perimeter minimizer** in Ω if

$$P_H(E;B_r(p)) \le P_H(F;B_r(p))$$

whenever $E \triangle F \Subset B_r(p) \subset \Omega$.

Measuring the 'flatness'

Assume $0 \in \partial E$ and $\nu_E(0) = \mathrm{e}_1$. We consider the vertical plane

$$\mathbb{W} = \{(x, y, t) \in \mathbb{H}^n : x_1 = 0\} = \mathbb{H}^{n-1} \times \mathbb{R}.$$

We define the **cylindrical excess** of E in C_r as

$$\mathrm{Exc}(E, C_r) = \frac{1}{r^{2n+1}} \int_{\partial E \cap C_r} \frac{|\nu_E - \nu_E(0)|_g^2}{2} \, d\mathcal{S}^{2n+1},$$

where

$$C_r = D_r * (-r, r) = \{ w * s\mathrm{e}_1 : w \in \mathbb{W},\ \|w\|_\infty < r,\ s \in (-r, r) \}.$$

The excess measures the 'flatness' of ∂E inside the cylinder.

Remark (Monti). Let $n \geq 2$. Then

$$\mathrm{Exc}(E, C_r) = 0 \text{ for some } r > 0 \implies E \cap C_r = \{p \in C_r : x_1 > 0\} \text{ and } \nu_E = \mathrm{e}_1$$

For $n = 1$ this property **fails**!

Intrinsic Lipschitz functions

We say that $f \in \mathrm{Lip}_H(\mathbb{W};\mathbb{R})$ with $\mathrm{Lip}_H(f) \le L$ if

$$|f(\pi_{\mathbb{W}}(p)) - f(\pi_{\mathbb{W}}(q))| \le L\|\pi_{\mathbb{W}}(q^{-1} * p)\|_\infty \quad \text{for all } p, q \in \mathrm{gr}_H(f),$$

where the **intrinsic graph** of f is

$$\mathrm{gr}_H(f) = \{\exp(f(w)X_1)(w) : w \in \mathbb{W}\} = \{w * f(w)\mathrm{e}_1 : w \in \mathbb{W}\}.$$

If $f \in \mathrm{Lip}_H(\mathbb{W};\mathbb{R})$, then a **non-linear intrinsic gradient** $\nabla^f f$ is well-defined:

$$\nabla^f f = (X_2 f, \dots, X_n f, \mathcal{B}f, Y_2 f, \dots, Y_n f)$$

where, in the sense of distributions, $\mathcal{B}f = \frac{\partial f}{\partial y_1} - 4f\frac{\partial f}{\partial t}$.

As in the Euclidean setting, one has

Theorem (Bigolin, Caravenna & Serra Cassano, 2015)

$$f \in \mathrm{Lip}_{H,loc}(\mathbb{W};\mathbb{R}) \iff \nabla^f f \in L^\infty_{loc}$$

Theorem (Citti, Manfredini, Pinamonti & Serra Cassano, 2014)

$$f \in \mathrm{Lip}_H(\mathbb{W};\mathbb{R}) \implies \mathcal{S}^{2n+1}(\mathrm{gr}_H(f|_K)) = c_n \int_K \sqrt{1 + |\nabla^f f|^2}\, d\mathcal{L}^{2n} \ \forall K \Subset \mathbb{W}$$

Height bound and (improved) Lipschitz approximation

The starting point is the control on the height of ∂E.

Height bound (Monti & Vittone, 2015) *[Maggi-like version]*

Let $n \geq 2$*. If* $E \subset \mathbb{H}^n$ *is H-minimizer in* C_{16} *with* $\mathcal{E} = \mathrm{Exc}(E, C_{16}) \leq \varepsilon_n$*, then*

$$\sup\{|x_1| : p \in C_1 \cap \partial E\} \leq c_n \, \mathcal{E}^{\frac{1}{2(2n+1)}}.$$

By this control on the height, one deduces the following result. This is an improved version of a previous approximation due to Monti (2014).

Theorem (Monti & S., 2016) *[Maggi-like version]*

Let $n \geq 2$*. If* $E \subset \mathbb{H}^n$ *is H-minimizer in* C_{5124}*,* $\mathcal{E} = \mathrm{Exc}(E, C_{5124}) \leq \varepsilon_n$ *and*

$$M = C_1 \cap \partial E, \qquad M_0 = \left\{ q \in M : \sup_{0<s<256} \mathrm{Exc}(E, C_s(q)) \leq \delta_n \right\},$$

then there exists $f \in \mathrm{Lip}_H(\mathbb{W}; \mathbb{R})$ *with* $\mathrm{Lip}_H(f) \leq 1$ *such that*

$$\sup_{\mathbb{W}} |f| \leq c_n \, \mathcal{E}^{\frac{1}{2(2n+1)}}, \qquad M_0 \subset M \cap \Gamma, \quad \Gamma = \mathrm{gr}_H(f|_{D_1})$$

$$\boxed{\mathcal{S}^{2n+1}(M \triangle \Gamma) \leq c_n \, \mathcal{E},} \qquad \int_{D_1} |\nabla^f f|^2 \, d\mathcal{L}^{2n} \leq c_n \, \mathcal{E}.$$

Approximation with estimate on the coincidence set

Adapting the strategy followed by De Lellis & Spadaro (2011) to study regularity of integer rectifiable currents in $\mathbb{R}^n$, we are able to estimate the 'size' of the set K where $\mathrm{gr}_H(f)$ and ∂E coincide.

Theorem (Monti & S., 2016) *[De Lellis & Spadaro-like version]*

Let $n \geq 2$, $\alpha \in (0, \frac{1}{2})$. If $E \subset \mathbb{H}^n$ is H-minimizer in C_k, $\mathcal{E} = \mathrm{Exc}(E, C_k) \leq \varepsilon_{\alpha,n}$, then there exist $K \subset D_1$ and $f \in \mathrm{Lip}_H(\mathbb{W}; \mathbb{R})$ such that

$$\boxed{\mathcal{L}^{2n}(D_1 \setminus K) \leq c_n\, \mathcal{E}^{1-2\alpha},} \qquad \mathrm{gr}_H(f|_K) = \partial E \cap (K * \mathbb{R}), \qquad \mathrm{Lip}_H(f) \leq c_n\, \mathcal{E}^{\alpha},$$

$$\boxed{\mathcal{S}^{2n+1}((\partial E \triangle \mathrm{gr}_H(f)) \cap C_1) \leq c_n\, \mathcal{E}^{1-2\alpha},} \qquad \int_{D_1} |\nabla^f f|^2 \, d\mathcal{L}^{2n} \leq c_n\, \mathcal{E}.$$

Although we can recover the correct estimates, K is **not optimal**: since the theory of currents in $\mathbb{H}^n$ is not yet well-established, we do not have the *BV estimate*.

However, one can refine the *Maggi-like* approximation following the strategy in Schoen & Simon (1982) to get an **optimal** K but with a **weaker** estimate.

What's next?

The next step in De Giorgi's proof is the **harmonic approximation**: given $r_k \to 0$, take $E_k = \delta_{1/r_k}(E)$ and study the limit ψ of the approximations $\varphi_k \in \mathrm{Lip}_H(\mathbb{W})$.

Theorem (Monti, 2015)

Let $n \geq 2$ and let $E \subset \mathbb{H}^n$ be H-minimizer in C_1. There exist $A \subset \mathbb{W}$ ngb of $0 \in \mathbb{W}$, constants $\bar{\varphi}_k \in \mathbb{R}$ and $\boxed{\psi \in W^{1,2}_{\mathbb{W}}(A)}$ such that, up to subseqsences, we have

$$\frac{\varphi_k - \bar{\varphi}_k}{\sqrt{\mathrm{Exc}(E_k, C_1)}} \rightharpoonup \psi \quad \text{and} \quad \frac{\nabla^{\varphi_k}\varphi_k}{\sqrt{\mathrm{Exc}(E_k, C_1)}} \rightharpoonup \nabla_{\mathbb{W}}\psi \quad \textit{weakly in } L^2.$$

*Moreover, $\boxed{\psi \in W^{1,2}_{\mathbb{W}}(A)}$ does **not** depend on the variable $y_1 \in \mathbb{R}$.*

Here $W^{1,2}_{\mathbb{W}}(A)$ is the space of all $\psi \in L^2(A)$ with distributional **linear** gradient

$$\nabla_{\mathbb{W}}\psi = \left(X_2\psi, \ldots, X_n\psi, \frac{\partial \psi}{\partial y_1}, Y_2\psi, \ldots, Y_n\psi\right) \in L^2(A; \mathbb{R}^{2n-1}).$$

Problem: prove the convergence is strong and ψ is harmonic, i.e. $-\Delta_{\mathbb{H}^{n-1}}\psi = 0$.

(This obstacle is also due to the *rigidity* of Korányi-Reimann deformations).

Thank you for your attention!

Extras: other approximations

Theorem (Monti, 2014)

Let $n \geq 2$. If $E \subset \mathbb{H}^n$ is H-minimizer in C_k and $\mathcal{E} = \mathrm{Exc}(E, C_k) \leq \varepsilon_n$, then there exists $f \in \mathrm{Lip}_H(\mathbb{W}; \mathbb{R})$ with $\mathrm{Lip}_H(f) \leq 1$ such that

$$\boxed{\mathcal{S}^{2n+1}((\partial E \triangle \mathrm{gr}_H(f)) \cap C_1) \leq c_n \, \mathcal{E}.}$$

Taking the excess in infinite cylinders $C_r^\infty = D_r * \mathbb{R}$, we can prove:

Theorem (S., 2016) *[Schoen & Simon-like version]*

Let $n \geq 2$, $\gamma \leq 1$. If $E \subset \mathbb{H}^n$ is H-minimizer in C_{129}^∞, $\mathcal{E} = \mathrm{Exc}(E, C_{129}^\infty) \leq \varepsilon_n$, then there exist $K \subset D_1$ and $f \in \mathrm{Lip}_H(\mathbb{W}; \mathbb{R})$ such that

$$\boxed{\mathcal{L}^{2n}(D_1 \setminus K) \leq c_n \, \gamma^{-2(2n+1)} \, \mathcal{E},} \quad \mathrm{gr}_H(f|_K) = \partial E \cap K * \mathbb{R},$$

$$\sup_{\mathbb{W}} |f| \leq c_n \, \mathcal{E}^{\frac{1}{2(2n+1)}}, \quad \mathrm{Lip}_H(f) \leq \gamma,$$

$$\boxed{\mathcal{S}^{2n+1}((\partial E \triangle \mathrm{gr}_H(f)) \cap C_1) \leq c_n \, \gamma^{-2(2n+1)} \, \mathcal{E},} \quad \int_{D_1} |\nabla^f f|^2 \, d\mathcal{L}^{2n} \leq c_n \gamma^2 \, \mathcal{E}.$$