ARCHIMEDES, A DINNER AND A THEOREM

A divertissement on the monotonicity of perimeter

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Abstract. If $A, B \subset \mathbb{R}^n$ are two convex bodies and $A \subset B$, then the perimeter of A does not exceed the perimeter of B. This monotonicity property of the perimeter dates back to the ancient Greek and Archimedes himself took it as a postulate in his celebrated work on the sphere and the cylinder. A few years ago, a couple of papers by M. Carozza, F. Giannetti, F. Leonetti, and A. Passarelli di Napoli established lower bounds on the difference of the perimeters of A and B in terms of their Hausdorff distance when $n = 2$ and $n = 3$. In this talk, after a brief introduction on the problem and the known results, I will generalise these lower bounds to any dimension $n \geq 4$. Time permitting, I will show how this approach can be extended to the case of anisotropic Wulff perimeters.

1. Archimedes and the Monotonicity of Perimeter

The ambient space is $\mathbb{R}^n$ with $n \geq 2$. For all $s \geq 0$ we let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure (in particular, $\mathcal{H}^0$ is the counting measure).

Definition 1.1. A *convex body* $E \subset \mathbb{R}^n$ is a compact convex set with non-empty interior.

If $E \subset \mathbb{R}^n$ is a k-dimensional convex body, with $1 \leq k \leq n$, we let ∂E be its boundary, which is a set of Hausdorff dimension $(k-1)$.

Definition 1.2. If $E \subset \mathbb{R}^n$ is a convex body, then $P(E) = \mathcal{H}^{n-1}(\partial E)$ denotes the *perimeter* of E.

Proposition 1.3 (Monotonicity)**.** *If $A \subset B \subset \mathbb{R}^n$ are convex bodies, then*

$$P(A) \leq P(B). \tag{1.1}$$

Inequality (1.1) is weel-known and dates back to the ancient Greek. Archimedes (287 b.C. – 212 b.C.) took it as a postulate in his work on the sphere and the cylinder, [1, p. 36]. Various proofs of (1.1) are possible: via the Cauchy formula for the area surface of convex bodies or by the monotonicity property of mixed volumes, [2, §7], by the Lipschitz property of the projection on a convex closed set, [3, Lemma 2.4], or by the fact that the perimeter is decreased under intersection with half-spaces, [7, Excercise 15.13].

Sketch of the proof of Proposition 1.3. Assume A has polyhedral boundary, so that $A = \bigcap_{i=1}^m H_m$, where $H_i = \{x \in \mathbb{R}^n : \langle x - p_i, \nu_i \rangle \geq 0\}$ is a closed half-space, with $p_i, \nu_i \in \mathbb{R}^n$, $|\nu_i| = 1$. Then it is enough to prove $P(B \cap H) \leq P(B)$ for any $H \subset \mathbb{R}^n$ closed half-space,

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since this easily implies

$$P(A) = P(B \cap A) = P\left(B \cap \bigcap_{i=1}^{m} H_m\right) \le P(B).$$

So let $H = \{x \in \mathbb{R}^n : \langle x - p, \nu \rangle \ge 0\}$ for some $p, \nu \in \mathbb{R}^n$, with $|\nu| = 1$. Define the constant vector field $X = -\nu$ on $\mathbb{R}^n$. Then, by the divergence theorem, we have

$$\begin{aligned} 0 &= \int_{B \cap H^c} \operatorname{div} X \; dx = \int_{\partial(B \cap H^c)} \left\langle X, \nu^{\mathrm{est}}_{B \cap H^c} \right\rangle \, dx \\ &= \int_{(\partial H) \cap B)} \langle X, \nu \rangle \; d\mathcal{H}^{n-1} + \int_{(\partial B) \cap H^c} \left\langle X, \nu^{\mathrm{est}}_{B} \right\rangle \, d\mathcal{H}^{n-1}. \end{aligned}$$

Thus $\mathcal{H}^{n-1}(\partial H \cap B) \le \mathcal{H}^{n-1}(\partial B \cap H^c)$ and so

$$\begin{aligned} P(B \cap H) &\le \mathcal{H}^{n-1}(\partial B \cap H) + \mathcal{H}^{n-1}(\partial H \cap B) \\ &\le \mathcal{H}^{n-1}(\partial B \cap H) + \mathcal{H}^{n-1}(\partial B \cap H^c) = P(B). \end{aligned}$$

If A is a convex body, then (by linear interpolation) we can find a sequence $(A_k)_{k \in \mathbb{N}}$ of convex body with polyhedral boundary such that $A_k \subset A$ and $P(A) = \lim_{k \to +\infty} P(A_k)$. Then $P(A_k) \le P(B)$ for all $k \in \mathbb{N}$ and the conclusion follows. □

Problem 1.4 (Converse)**.** *Under which assumptions on $E \subset \mathbb{R}^n$ the following implication*

$$P(E) \le P(C) \ \forall C \subset \mathbb{R}^n \text{ convex body}, \ E \subset C \quad \Longrightarrow \quad E \text{ convex body}$$

is true?

2. Lower Bounds and Leonetti's Dinner Problem

Since A and B are compact sets and $A \subset B$, the *Hausdorff distance* of A and B is

$$h(A, B) = \max_{y \in B} \min_{x \in A} |x - y|. \tag{2.1}$$

Let $a \in A$ and $b \in B$ be such that $h(A, B) = |a - b|$. It turns out that $b \in B \setminus A$ and a is the orthogonal projection of b onto the closed convex set A.

Lower bounds for the deficit $\delta(B, A) = P(B) - P(A)$ with respect to $h(A, B)$ of A and B have been recently established for $n = 2, 3$ in [4–6].

The case $n = 2$ was treated for the first time in [6], and was subsequently improved in [4] to the following inequality

$$P(A) + \frac{2h(A, B)^2}{\sqrt{\left(\frac{\mathcal{H}^1(B \cap L)}{2}\right)^2 + h(A, B)^2} + \frac{\mathcal{H}^1(B \cap L)}{2}} \le P(B), \tag{2.2}$$

where $L = \{x \in \mathbb{R}^2 : \langle b - a, x - a \rangle = 0\}$, see Figure 1.

The case $n = 3$ was studied in [5], where the authors proved the following inequality

$$P(A) + \frac{\pi d h(A, B)^2}{\sqrt{d^2 + h(A, B)^2} + d} \le P(B), \tag{2.3}$$

where $d = \operatorname{dist}(a, \partial B \cap \partial H)$ and $H = \{x \in \mathbb{R}^3 : \langle b - a, x - a \rangle \le 0\}$, see Figure 2.

Inequalities (2.2) and (2.3) are sharp, in the sense that they are equalities at least in one case, see [4, 5]. Inequality (2.3), however, does not seem to be the correct generalization of

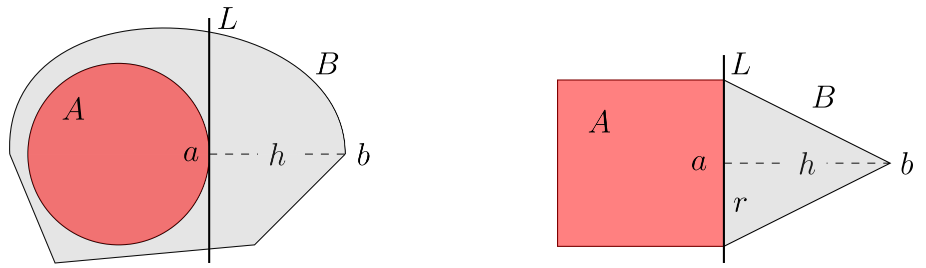

FIGURE 1. Inequality (2.2): setting (left) and optimal configuration (right).

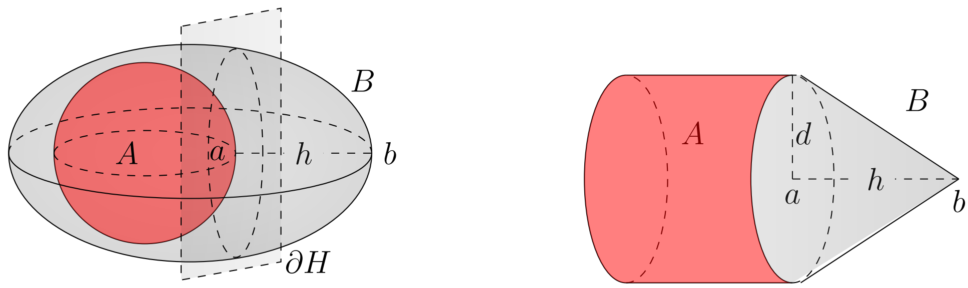

FIGURE 2. Inequality (2.3): setting (left) and optimal configuration (right).

inequality (2.2) to the case $n = 3$, because of the distance $d = \operatorname{dist}(a, \partial B \cap \partial H)$ replacing the bigger radius $r = \sqrt{\mathcal{H}^2(B \cap \partial H)/\pi}$.

Problem 2.1 (F. Leonetti's dinner problem, Levico Terme, January 2016)**.** *Is it possible to prove similar inequalities for $n \geq 4$?*

Theorem 2.2 ([9, Corollary 1.3])**.** *Let $n \geq 2$. If $A \subset B$ are two convex bodies in $\mathbb{R}^n$, then*

$$P(A) + \frac{\omega_{n-1} r^{n-2} h^2}{\sqrt{h^2 + r^2} + r} \leq P(B), \tag{2.4}$$

where $h = h(A, B)$ is the Hausdorff distance of A and B and

$$r = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(B \cap \partial H)}{\omega_{n-1}}}, \qquad H = \{x \in \mathbb{R}^n : \langle b - a, x - a \rangle \leq 0\}, \tag{2.5}$$

with $a \in A$ and $b \in B$ such that $|a - b| = h(A, B)$.

Inequality (2.4) is sharp, as one can easily check generalizing the examples given in Figures 1 and 2 to higher dimensions.

Problem 2.3 (Upper bounds)**.** *Prove that, if $A \subset B \subset \mathbb{R}^2$ are convex bodies, then $\delta(B, A) \leq 2\pi h(A, B)$. Does a similar upper bound hold for $n \geq 3$?*

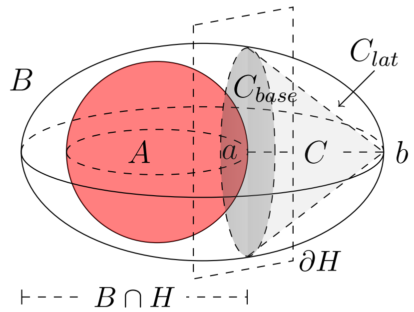

FIGURE 3. Setting of the proof of Theorem 2.2.

3. Proof of Theorem 2.2

Lemma 3.1 (Schwartz symmetrization). *Let $n \geq 2$ and let $E \subset \mathbb{R}^n$ be convex body. Define*

$$E^{Sch} := \left\{ x = (x', t) \in \mathbb{R}^n : |x'| \leq \left(\frac{\mathcal{H}^{n-1}(E_t)}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \right\}.$$

Then E^{Sch} is a convex body and $P(E^{\mathrm{Sch}}) \leq P(E)$.

Sketch of the proof of Theorem 2.2. Let $a \in A$ and $b \in B$ be such that $h(A, B) = |a - b|$ as in (2.1). By definition of the half-space H in (2.5) and by minimality of the projection, the closed hyperplane

$$\partial H = \{x \in \mathbb{R}^n : \langle b - a, x - a \rangle = 0\}$$

is a supporting one for the convex set A in the point a. We let $C = \mathcal{C}(b, B \cap \partial H)$ be the cone with vertex b and base $C_{\mathrm{base}} = B \cap \partial H$. Note that the lateral surface of C is given by $C_{\mathrm{lat}} = \mathcal{C}(b, \partial B \cap \partial H)$.

Since $A \subset B \cap H$, $B \cap H \subset B$ and $C \subset B \cap \overline{H^c}$, by the monotonicity formula (1.1) we have

$$P(A) \leq P(B \cap H) \leq P(B)$$

and therefore

$$\begin{aligned} \delta(B, A) &= \delta(B, B \cap H) + \delta(B \cap H, A) \\ &\geq \delta(B, B \cap H) = P(B) - P(B \cap H) \\ &= \mathcal{H}^{n-1}(\partial B \cap H^c) - \mathcal{H}^{n-1}(B \cap \partial H) \\ &\geq \mathcal{H}^{n-1}(C_{\mathrm{lat}}) - \mathcal{H}^{n-1}(C_{\mathrm{base}}). \end{aligned} \tag{3.1}$$

To conclude, we now just need to solve the *minimization problem*

$$\min\Big\{ \mathcal{H}^{n-1}(C_{\mathrm{lat}}) : C \text{ cone with given height } h \text{ and given base area } \mathcal{H}^{n-1}(C_{\mathrm{base}}) \Big\}. \tag{3.2}$$

We apply Lemma 3.1 to the cone C (up to a rotation, since we need to slice perpendicularly to its height). Then we immediately get that

$$\mathcal{H}^{n-1}((C^{\mathrm{Sch}})_{\mathrm{base}}) = \mathcal{H}^{n-1}(C_{\mathrm{base}}), \qquad \mathcal{H}^{n-1}((C^{\mathrm{Sch}})_{\mathrm{lat}}) \leq \mathcal{H}^{n-1}(C_{\mathrm{lat}}).$$

In particular, C^{Sch} is a right circular cone with

$$\mathcal{H}^{n-1}((C^{\mathrm{Sch}})_{\mathrm{base}}) = \omega_{n-1} r^{n-1}, \qquad \mathcal{H}^{n-1}((C^{\mathrm{Sch}})_{\mathrm{lat}}) = \omega_{n-1} r^{n-2} \sqrt{h^2 + r^2},$$

where r is the radius defined in (2.5). This concludes the proof. □

Problem 3.2 (Avoiding Lemma 3.1)**.** *Solve the minimization problem* (3.2) *for* $n = 2$ *without using Lemma 3.1. Can you solve it without using Lemma 3.1 also for* $n \geq 3$*?*

4. Monotonicity of Wulff perimeter

Inequality (1.1) naturally generalizes to the anisotropic (Wulff) perimeter. Precisely, given a positively 1-homogeneous convex function $\Phi\colon \mathbb{R}^n \to [0,\infty)$, if $A \subset B$ are two convex bodies in $\mathbb{R}^n$, then

$$P_\Phi(A) \leq P_\Phi(B). \tag{4.1}$$

Here $P_\Phi(E)$ denotes the *anisotropic* Φ*-perimeter* of a convex body $E \subset \mathbb{R}^n$ and is defined as

$$P_\Phi(E) = \int_{\partial E} \Phi(\nu_E)\, d\mathcal{H}^{n-1},$$

where $\nu_E\colon \partial E \to \mathbb{R}^n$ is the inner unit normal of E (defined $\mathcal{H}^{n-1}$-a.e. on ∂E). Clearly, when $\Phi(x) = |x|$ for all $x \in \mathbb{R}^n$, $P_\Phi(E) = \mathcal{H}^{n-1}(\partial E)$, the Euclidean perimeter of E. The Φ-perimeter obeys the scaling law $P_\Phi(\lambda E) = \lambda^{n-1} P_\Phi(E)$, $\lambda > 0$, and it is invariant under translations. However, at variance with the Euclidean perimeter, P_Φ is not invariant by the action of $O(n)$, or even of $SO(n)$, and in fact it may even happen that $P_\Phi(E) \neq P_\Phi(\mathbb{R}^n \setminus E)$, provided that Φ is not symmetric with respect to the origin.

Similarly to inequality (1.1), inequality (4.1) is a consequence of the Cauchy formula for the anisotropic perimeter or of the monotonicity property of mixed volumes, [2, §7, §8], or of the fact that the anisotropic perimeter is decreased under intersection with half-spaces, [7, Remark 20.3].

We conclude this note stating a lower bound for the anisotropic deficit $\delta_\Phi(B,A) = P_\Phi(B) - P_\Phi(A)$ with respect to the Hausdorff distance $h(A,B)$ of A and B. To do so, we need some preliminaries. Here and in the following, we let

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}, \qquad \nu^\perp = \{x \in \mathbb{R}^n : \langle x, \nu\rangle = 0\} \quad \forall \nu \in \mathbb{S}^{n-1}.$$

If Φ is positively 1-homogeneous, convex and coercive on $\mathbb{R}^n$, i.e. $\Phi(x) > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$, then Φ is admissible, since the choice $\phi_\nu(z) = |z|$, $z \in \nu^\perp$, and $g_\nu(s,t) = c\sqrt{s^2+t^2}$, $s,t \geq 0$, with $c = \min\{\Phi(x) : |x| = 1\}$, is possible for all $\nu \in \mathbb{S}^{n-1}$ (although not the best one for special directions in general).

Definition 4.1 (Admissible Φ)**.** Let $n \geq 2$ and let $\Phi\colon \mathbb{R}^n \to [0,\infty)$ be a positively 1-homogeneous convex function. We say that Φ is *admissible* if, for each $\nu \in \mathbb{S}^{n-1}$, there exist two functions $g_\nu\colon [0,\infty)^2 \to [0,\infty)$ and $\phi_\nu\colon \nu^\perp \to [0,\infty)$ such that

(i) g_ν is non-constantly zero, positively 1-homogeneous, convex and $s \mapsto g_\nu(s,t)$ is non-decreasing for each fixed $t \in [0,\infty)$;

(ii) ϕ_ν is positively 1-homogeneous, convex and coercive on $\nu^\perp$, i.e. $\phi_\nu(z) > 0$ for all $z \in \nu^\perp$, $z \neq 0$;

(iii) for all $x \in \mathbb{R}^n$ with $\langle x,\nu\rangle \geq 0$, it holds

$$\Phi(x) \geq g_\nu(\phi_\nu(x - \langle x,\nu\rangle\nu), \langle x,\nu\rangle).$$

We can now state our result, which is contained in the following theorem. In the sequel, for each $\nu \in \mathbb{S}^{n-1}$, we let $W_\nu \subset \nu^\perp$ be the Wulff shape associated with ϕ_ν in $\nu^\perp$, i.e.

$$W_\nu = \Big\{ z \in \nu^\perp : \phi_\nu^*(z) \le 1 \Big\}, \tag{4.2}$$

where $\phi_\nu^* \colon \nu^\perp \to [0, \infty)$ is given by $\phi_\nu^*(z) = \sup\{\langle z, w\rangle : \phi_\nu(w) < 1\}$ for all $z \in \nu^\perp$. Moreover, for any $a \in \mathbb{R}$ we let $a^+ = \max\{a, 0\}$.

Theorem 4.2 ([9, Theorem 1.2])**.** *Let $n \ge 2$ and let $\Phi \colon \mathbb{R}^n \to [0, \infty)$ be a positively* 1*-homogeneous convex function which is admissible in the sense of Definition 4.1. If $A \subset B$ are two convex bodies in $\mathbb{R}^n$, then*

$$P_\Phi(A) + \mathcal{H}^{n-1}(W_{\nu_H}) r^{n-2} \Big(g_{\nu_H}(h, r) - \Phi(\nu_H) r \Big)^+ \le P_\Phi(B), \tag{4.3}$$

where $h = h(A, B)$ is the Hausdorff distance of A and B and

$$r = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(B \cap \partial H)}{\mathcal{H}^{n-1}(W_{\nu_H})}}, \qquad H = \{x \in \mathbb{R}^n : \langle b - a, x - a\rangle \le 0\}, \qquad \nu_H = \frac{a - b}{|a - b|}, \tag{4.4}$$

with $a \in A$ and $b \in B$ such that $|a - b| = h(A, B)$.

Open Problem 4.3 (Carnot groups)**.** *Let $\mathbb{G}$ be a Carnot group on $\mathbb{R}^n$ and let $P_{\mathbb{G}}$ be horizontal perimeter in $\mathbb{G}$. It is known that, if $A \subset B$ are $\mathbb{G}$-convex bodies in $\mathbb{G}$ (see* [8, Definition 3.15] *for a definition), then $P_{\mathbb{G}}(A) \le P_{\mathbb{G}}(B)$, see* [8, Corollary 3.20]*. Is it possible to prove an analogous version of Theorem 2.2 in this setting?*

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