Heat and entropy flows in Carnot groups

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Sketchy idea: Otto's calculus

Take $(v_t)_{t>0}\colon \mathbb{R}^n \to \mathbb{R}^n$ time-dependent vector field. Consider the solution of

$$\partial_t u_t + \mathrm{div}(v_t u_t) = 0 \qquad \text{in } \mathbb{R}^n \times (0,+\infty). \tag{CE}$$

In Otto's calculus, v_t is the "velocity" of $\mu_t = u_t \mathscr{L}^n \in \mathscr{P}(\mathbb{R}^n)$ for $t > 0$.

The (Shannon-Boltzmann) entropy

$$\mathrm{Ent}(\mu_t) := \int_{\mathbb{R}^n} u_t \log u_t \, dx$$

along the curve $(\mu_t)_{t>0}$ satisfies

$$\frac{d}{dt}\mathrm{Ent}(\mu_t) \overset{(CE)}{=} -\int_{\mathbb{R}^n} (1+\log u_t)\, \mathrm{div}(v_t u_t)\, dx = \int_{\mathbb{R}^n} \left\langle \frac{\nabla u_t}{u_t}, v_t \right\rangle d\mu_t.$$

We say that $(\mu_t)_{t>0}$ is a gradient flow of Ent if $t \mapsto \mathrm{Ent}(\mu_t)$ has maximal dissipation rate. Having in mind "$\frac{d}{dt}\mathrm{Ent}(\mu_t) = \nabla \mathrm{Ent}(\mu_t) \cdot v_t$", we deduce

$$(\mu_t)_{t>0} \text{ gradient flow of Ent} \iff v_t = -\frac{\nabla u_t}{u_t} \iff \partial_t u_t = \Delta u_t.$$

Problem: prove this equivalence (rigorously!) in $(X, \mathsf{d}, \mathfrak{m})$.

Let $(X, \mathbf{d})$ be a metric space. A curve $\gamma : I \subset \mathbb{R} \to X$ is $AC^p(I; (X, \mathbf{d}))$ if

$$\exists g \in L^p(I) \qquad \text{such that} \qquad \mathbf{d}(\gamma_s, \gamma_t) \leq \int_s^t g(r)\, dr \qquad \forall s, t \in I,\ s < t.$$

The minimal $g \in L^p(I)$ is the metric derivative $|\dot{\gamma}_t| = \lim_{s \to t} \frac{\mathbf{d}(\gamma_s, \gamma_t)}{|s-t|}$.

Let $E : X \to \mathbb{R} \cup \{+\infty\}$ be a function. A metric gradient flow of E starting from $\gamma_0 \in \mathrm{Dom}(E)$ is a curve $\gamma \in AC^1_{loc}([0, +\infty); (X, \mathbf{d}))$ such that

$$E(\gamma_t) + \frac{1}{2} \int_s^t |\dot{\gamma}_r|^2\, dr + \frac{1}{2} \int_s^t |D^- E|^2(\gamma_r)\, dr \leq E(\gamma_s) \qquad \forall t > s \geq 0. \quad \text{(EDI)}$$

Here $|D^- E|(x) = \limsup_{y \to x} \max\left\{ \frac{E(x) - E(y)}{\mathbf{d}(x,y)}, 0 \right\}$ is the descending slope of E.

If X is a Hilbert space, then $\dot{\gamma}_t = -\nabla E(\gamma_t)$ and so

$$\frac{d}{dt} E(\gamma_t) \overset{\text{(chain)}}{=} \langle \nabla E(\gamma_t), \dot{\gamma}_t \rangle \overset{\text{(CS)}}{=} -|\nabla E(\gamma_t)| \cdot |\dot{\gamma}_t| \overset{\text{(Y)}}{=} -\frac{1}{2} |\nabla E(\gamma_t)|^2 - \frac{1}{2} |\dot{\gamma}_t|^2.$$

Let (X, d) be a Polish (geodesic) metric space. We endow the set

$$\mathscr{P}_2(X) = \left\{ \mu \in \mathscr{P}(X) : \int_X \mathsf{d}(x, x_0)^2 \, d\mu(x) < +\infty, \; x_0 \in X \right\}$$

with the Wasserstein distance: for any $\mu, \nu \in \mathscr{P}(X)$, we set

$$W_2^2(\mu, \nu) = \inf \left\{ \int_{X \times X} \mathsf{d}(x, y)^2 \, d\pi : \pi \in \Gamma(\mu, \nu) \right\},$$

where $\Gamma(\mu, \nu) = \{\pi \in \mathscr{P}(X \times X) : (p_1)_\# \pi = \mu, \; (p_2)_\# \pi = \nu\}$.

Property: $(\mathscr{P}_2(X), W_2)$ is a Polish (geodesic) metric space.

Let $\mathfrak{m}$ be a non-negative, σ-finite Borel measure on X such that

$$\mathfrak{m}\left(\{x \in X : d(x, x_0) < r\}\right) \leq Ae^{Br^2} \qquad \exists A, B > 0. \qquad \text{(exp.ball)}$$

The entropy functional $\mathrm{Ent}_{\mathfrak{m}} : \mathscr{P}_2(X) \to (-\infty, +\infty]$ is defined as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) = \begin{cases} \int_X \rho \log \rho \, d\mathfrak{m} & \text{if } \mu = \rho \mathfrak{m} \in \mathscr{P}_2(X), \\ +\infty & \text{otherwise.} \end{cases}$$

Assumption (exp.ball) ensures that $\mathrm{Ent}(\mu) > -\infty \; \forall \mu \in (\mathscr{P}_2(X), W_2)$.

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. The Dirichlet-Cheeger energy of $u\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is given by

$$\mathrm{Ch}(u) = \inf\left\{\liminf_{n} \int_X |Du_n|^2 \, d\mathfrak{m} : u_n \to u \text{ in } L^2(X, \mathsf{d}, \mathfrak{m}),\ u_n \in \mathrm{Lip}(X; \mathbb{R})\right\}.$$

Here $|Du|(x) = \limsup_{y \to x} \frac{|u(y)-u(x)|}{\mathsf{d}(x,y)}$ denotes the slope of $u \in \mathrm{Lip}(X; \mathbb{R})$.

Properties: Dirichlet-Cheeger energy is convex and l.s.c.

The heat flow in $(X, \mathsf{d}, \mathfrak{m})$ is the (Hilbertian) gradient flow of Ch in $L^2(X, \mathfrak{m})$: for $u_0 \in L^2(X, \mathfrak{m})$, $\exists t \mapsto u_t = H_t(u_0) \in \mathrm{Lip}_{loc}((0, +\infty); L^2(X, \mathfrak{m}))$ such that

$$u_t \xrightarrow[t \to 0]{} u_0 \text{ in } L^2(X, \mathfrak{m}) \qquad \text{and} \qquad \frac{d}{dt} u_t \in -\partial^- \mathrm{Ch}(u_t) \text{ for a.e. } t > 0.$$

We set $-\Delta u \in \partial^- \mathrm{Ch}(u)$ the element of minimal $L^2(X, \mathfrak{m})$-norm.

Be careful: $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $\|u\|_{W^{1,2}} = \left(\|u\|_2^2 + \|\mathrm{Ch}(u)\|_2^2\right)^{1/2}$ is Banach, but not Hilbert in general! Example: $(\mathbb{R}^n, \|\cdot\|_p, \mathscr{L}^n)$ for $p \neq 2$.

CD$(K,+\infty)$ metric measure spaces

The space $(X,\mathsf{d},\mathfrak{m})$ is CD$(K,+\infty)$ for some $K\in\mathbb{R}$ if $\forall \mu_0,\mu_1\in \mathrm{Dom(Ent)}$ $\exists \mu_t : [0,1]\to \mathscr{P}(X)$ constant speed geodesic joining μ_0 and μ_1 such that

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \le (1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}t(1-t)W_2^2(\mu_0,\mu_1). \qquad (K)$$

The space $(X,\mathsf{d},\mathfrak{m})$ is RCD$(K,+\infty)$ if, in addition, $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ is Hilbertian.

Theorem (von Renesse - Sturm, 2005)

Let (M,g) be a Riemannian manifold. Then *(K)* holds if and only if $\mathrm{Ric}\ge K$.

Theorem (Gigli, 2010; Ambrosio - Gigli - Savaré, 2014)

Assume $(X,\mathsf{d},\mathfrak{m})$ is *(exp.ball)* and CD$(K,+\infty)$. Consider

- $u_t = H_t(u_0)$ a GF of Ch in $L^2(X,\mathfrak{m})$ starting from $u_0\in L^2(X,\mathfrak{m})$;
- μ_t a GF of Ent in $(\mathscr{P}_2(X),W_2)$ starting from $\mu_0 = u_0\mathfrak{m}\in \mathrm{Dom(Ent)}$.

Then the two GFs are unique and coincide, i.e. $\mu_t = u_t\mathfrak{m}$ for all $t>0$.

- X Riemannian manifold with $\mathrm{Ric}\ge K$ (Erbar, 2010);
- X Alexandrov space (Gigli - Kuwada - Otha, 2009 - 2013).

A Carnot group $\mathbb{G}$ is a connected, simply connected, stratified Lie group with

$$\mathbf{Lie}(\mathbb{G}) = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \quad V_i = [V_1, V_{i-1}], \quad [V_1, V_\kappa] = \{0\}.$$

By Campbell-Hausdorff formula, $\mathbb{G} \sim (\mathbb{R}^n, \cdot)$ using exponential coordinates.
We call $H\mathbb{G} = V_1$ the horizontal directions. If $V_1 = \text{span}\{X_1, \dots, X_m\}$, then $\nabla_{\mathbb{G}} u = \sum_{j=1}^m (X_j u) X_j \in V_1$ and $\Delta_{\mathbb{G}} u = \sum_{j=1}^m X_j^2 u$ (Kohn's Laplacian).
The Carnot-Carathéodory distance of $x, y \in \mathbb{G}$ is

$$\mathbf{d}_{cc}(x, y) = \inf\left\{ \int_0^1 \|\dot\gamma_t\|_{\mathbb{G}}\, dt : \gamma \in \mathrm{Lip}([0,1]; \mathbb{R}^n),\ \gamma_0 = x,\ \gamma_1 = y,\ \dot\gamma_t \in V_1 \right\}.$$

Then $(\mathbb{G}, \mathbf{d}_{cc}, \mathscr{L}^n)$ is Polish, geodesic and $\mathscr{L}^n(\mathbf{B}_{cc}(x, r)) = Cr^Q$, $Q \in \mathbb{N}$.

Proposition: the space $(\mathbb{G}, \mathbf{d}_{cc}, \mathscr{L}^n)$ is not CD$(K, +\infty)$ for any $K \in \mathbb{R}$!

Proof. By contradiction, assume $(\mathbb{G}, \mathbf{d}_{cc}, \mathscr{L}^n)$ is CD$(K, +\infty)$ $\exists K \in \mathbb{R}$:
$\Rightarrow$ $(\mathbb{G}, \mathbf{d}_{cc}, \mathscr{L}^n)$ is RCD$(K, +\infty)$ [AGMR, 2015]
$\Rightarrow$ $(\mathbb{G}, \mathbf{d}_{cc}, \mathscr{L}^n)$ is BE$(K, +\infty)$, i.e. $\|\nabla_{\mathbb{G}} H_t(u)\|^2_{\mathbb{G}} \le e^{-2Kt} H_t(\|\nabla_{\mathbb{G}} u\|^2_{\mathbb{G}})$ [ibid.]
$\Rightarrow$ contradiction to $\|\nabla_{\mathbb{G}} H_t(u)\|^2_{\mathbb{G}} \le \Gamma_{\mathbb{G}} H_t(\|\nabla_{\mathbb{G}} u\|^2_{\mathbb{G}})$ $\forall t \ge 0$ [Melcher, 2007]

Non-CD$(K, +\infty)$ spaces: Heisenberg groups

The Heisenberg group $\mathbb{H}^n$ is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the group law

$$(x, y, u) \cdot (x', y', u') = \left(x + x', y + y', u + u' - \frac{1}{2} \sum_{k=1}^{n} (x_k y'_k - x'_k y_k) \right).$$

The tangent bundle is given by

$$X_k = \partial_{x_k} - \frac{y_k}{2} \partial_u, \quad Y_k = \partial_{y_k} + \frac{x_k}{2} \partial_u, \quad U = [X_k, Y_k] = \partial_u, \quad k = 1, \dots, n.$$

Here $\kappa = 2$, $H\mathbb{H}^n = \operatorname{span}\{X_k, Y_k : k \le n\}$, $\Delta_{\mathbb{H}^n} = \sum_{k=1}^n X_k^2 + Y_k^2$, $Q = 2n + 2$.

Theorem (Juillet, 2014)

The space $(\mathbb{H}^n, \mathsf{d}_{cc}, \mathscr{L}^{2n+1})$ is not CD$(K, +\infty)$ [Juillet, 2009]. Consider

- u_t solution of $\partial_t u_t = \Delta_{\mathbb{H}^n} u_t$ with initial datum $u_0 \in L^1(\mathbb{H}^n, \mathscr{L}^{2n+1})$;
- μ_t a GF of Ent in $(\mathscr{P}_2(\mathbb{H}^n), W_2)$ from $\mu_0 = u_0 \mathscr{L}^{2n+1} \in \operatorname{Dom}(\mathrm{Ent})$.

Then the two GFs are unique and coincide, i.e. $\mu_t = u_t \mathscr{L}^{2n+1}$ for all $t > 0$.

Problem [Juillet, 2014]: is it true in any Carnot group? YES!

Correspondence of the two GFs in Carnot groups

Theorem (Ambrosio - S., 2018)

Let $(\mathbb{G}, \mathsf{d}_{cc}, \mathscr{L}^n)$ be a Carnot group. Consider

- u_t solution of $\partial_t u_t = \Delta_{\mathbb{G}} u_t$ with initial datum $u_0 \in L^1(\mathbb{G}, \mathscr{L}^n)$;
- μ_t a GF of Ent in $(\mathscr{P}_2(\mathbb{G}), W_2)$ starting from $\mu_0 = u_0 \mathscr{L}^n \in \mathrm{Dom}(\mathrm{Ent})$.

Then the two GFs are unique and coincide, i.e. $\mu_t = u_t \mathscr{L}^n$ for all $t > 0$.

Strategy of proof

- GF heat $\Rightarrow$ GF Ent:
  - Riemannian approximation of $\mathbb{G}$
  - global estimates on heat kernel
  - entropy dissipation along GF heat
- GF Ent $\Rightarrow$ GF heat:
  - smoothing in time variable
  - smoothing in space variable (group structure!)
  - continuity equation for smoothed curve

Riemannian approximation of $\mathbb{G}$

For $\varepsilon > 0$ consider $(\mathbb{G}_\varepsilon, g_\varepsilon, \mathrm{vol}_\varepsilon)$ the manifold $\mathbb{R}^n$ endowed with the metric

$$\left\langle \varepsilon^{d(i)-1} X_i, \varepsilon^{d(j)-1} X_j \right\rangle_\varepsilon = \delta_{ij}, \qquad \forall X_i \in V_{d(i)}, \quad i = 1, \dots, n.$$

Example: for Heisenberg group $\mathbb{H}^1 \sim \mathbb{R}^3$ consider

$$[X, Y] = U, \quad \langle X, Y \rangle_\varepsilon = \langle X, U \rangle_\varepsilon = \langle Y, U \rangle_\varepsilon = 0 \quad \|X\|_\varepsilon = \|Y\|_\varepsilon = \|\varepsilon U\|_\varepsilon = 1.$$

Properties: $\mathsf{d}_\varepsilon \leq \mathsf{d}_{cc}$ are left-invariant and $\mathrm{vol}_\varepsilon = \varepsilon^{n-Q} \mathscr{L}^n$.

In particular $\mathrm{Ent}_{\mathrm{vol}_\varepsilon}(\mu) = \mathrm{Ent}_{\mathscr{L}^n}(\mu) + \log(\varepsilon^{Q-n})$ for all $\varepsilon > 0$

$\Rightarrow$ we consider $(\mathbb{G}_\varepsilon, \mathsf{d}_\varepsilon, \mathscr{L}^n)$, i.e. $(\mathbb{G}_\varepsilon, \mathsf{d}_\varepsilon, \mathrm{vol}_\varepsilon)$ with rescaled volume.

The ε-Riemannian gradient is $\nabla_\varepsilon u = \sum_{i=1}^{\kappa} \varepsilon^{2(i-1)} \nabla_{V_i} u$.

Key fact: Riemannian variety with $\mathrm{Ric} \geq -K_{\mathbb{G}} \varepsilon^{-2}$ for some $K_{\mathbb{G}} > 0$

$\Rightarrow$ we can use [Erbar, 2010] in the Riemannian approximation!

Fact: $(\mathbb{G}_\varepsilon, g_\varepsilon, \mathrm{vol}_\varepsilon) \longrightarrow (\mathbb{G}, \mathsf{d}_{cc}, \mathscr{L}^n)$ in pointed Gromov-Hausdorff sense.

Properties of the heat kernel

Assume $\partial_t u_t = \Delta_{\mathbb{G}} u_t$ with initial datum $u_0 \in L^1(\mathbb{G}, \mathscr{L}^n)$. Then

$$u_t(x) = u_0 \star \mathbf{h}_t(x) = \int_{\mathbb{G}} \mathbf{h}_t(y^{-1}x)\, u_0(y)\, dy.$$

Theorem (Varopoulos, Saloff-Coste, Coulhon)

(i) $\exists C = C(\mathbb{G}) > 0$ such that

$$\mathbf{h}_t(x) \le C t^{-Q/2} \exp\left(-\frac{\mathbf{d}_{cc}(x,0)^2}{4t}\right)$$

(ii) $\forall \varepsilon > 0\ \exists C_\varepsilon > 0$ such that

$$\mathbf{h}_t(x) \ge C_\varepsilon t^{-Q/2} \exp\left(-\frac{\mathbf{d}_{cc}(x,0)^2}{4(1-\varepsilon)t}\right)$$

(iii) $\forall j \in \mathbb{N}\ \forall \varepsilon > 0\ \exists C_\varepsilon(j,l) > 0$ such that

$$|X_{i_1} \cdots X_{i_j} \mathbf{h}_t(x)| \le C_\varepsilon(j,l) t^{-\frac{Q+j}{2}} \exp\left(-\frac{\mathbf{d}_{cc}(x,0)^2}{4(1+\varepsilon)t}\right)$$

where $X_{i_1} \cdots X_{i_j} \in V_1$.

Continuity equation in $(\mathscr{P}_2(X), W_2)$

Let $I \subset \mathbb{R}$ be an open interval. For $v^{\mathbb{G}} : I \times \mathbb{G} \to H\mathbb{G}$ such that

$$\int_I \int_{\mathbb{G}} \|v_t^{\mathbb{G}}(x)\|_{\mathbb{G}} \, d\mu_t(x) \, dt < +\infty$$

and $t \mapsto \mu_t \in \mathscr{P}_2(\mathbb{G})$ we study $\partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0$, i.e.

$$\int_I \int_{\mathbb{G}} \partial_t \varphi(t,x) + \left\langle v_t^{\mathbb{G}}(x), \nabla_{\mathbb{G}} \varphi(t,x) \right\rangle_{\mathbb{G}} d\mu_t(x) dt = 0 \qquad \forall \varphi \in C_c^\infty. \quad \text{(CE)}$$

Proposition (Gigli-Han, 2015; Juillet, 2014; Ambrosio-S., 2018)

(i) $\mu_t \in AC^2_{loc}(I; \mathscr{P}_2(\mathbb{G})) \Rightarrow \|v_t^{\mathbb{G}}\|_{L^2_{\mathbb{G}}(\mu_t)} \in L^2_{loc}(I)$ such that (CE) holds and

$$v_t^{\mathbb{G}} \in \mathrm{Tan}_{\mathbb{G}}(\mu_t), \qquad \|v_t^{\mathbb{G}}\|_{L^2_{\mathbb{G}}(\mu_t)} = |\dot{\mu}_t|_{\mathbb{G}} \quad \text{for a.e. } t \in I$$

(ii) $\mu_t \subset \mathscr{P}_2(\mathbb{G})$ satisfies (CE) with $\|v_t^{\mathbb{G}}\|_{L^2_{\mathbb{G}}(\mu_t)} \in L^2_{loc}(I) \Rightarrow \mu_t \in AC^2_{loc}$ and

$$|\dot{\mu}_t|_{\mathbb{G}} \le \|v_t^{\mathbb{G}}\|_{L^2_{\mathbb{G}}(\mu_t)} \qquad \text{for a.e. } t \in I.$$

Proof. True in $(\mathbb{G}_\varepsilon, \mathbf{d}_\varepsilon, \mathscr{L}^n)$ by [Erbar] $\Rightarrow$ pass to the limit as $\varepsilon \to 0$.

Fisher information and descending slope of Ent

The Fisher information of $\mu = \rho\mathscr{L}^n \in \mathscr{P}_2(\mathbb{G})$ is

$$\mathsf{F}_{\mathbb{G}}(\rho) = \int_{\mathbb{G}\cap\{\rho>0\}} \frac{\|\nabla_{\mathbb{G}}\rho\|_{\mathbb{G}}^2}{\rho} \, \mathrm{d}\mathscr{L}^n.$$

Proposition (Ambrosio-Gigli-Savaré, 2014)

If $|\mathsf{D}_{\mathbb{G}}^{-}\mathsf{Ent}|(\mu) < \infty$, then $\mathsf{F}_{\mathbb{G}}(\rho) \le |\mathsf{D}_{\mathbb{G}}^{-}\mathsf{Ent}|^2(\mu)$.

Proof. This is a general fact, true in $(\mathsf{X}, \mathbf{d}, \mathfrak{m})$!

Proposition (Juillet, 2014; Ambrosio - S., 2018)

If $|\mathsf{D}_{\varepsilon}^{-}\mathsf{Ent}|(\mu) < \infty$, then $|\mathsf{D}_{\mathbb{G}}^{-}\mathsf{Ent}|(\mu) < \infty$ and $\mathsf{F}_{\mathbb{G}}(\rho) = |\mathsf{D}_{\mathbb{G}}^{-}\mathsf{Ent}|^2(\mu)$.

Proof. True in $(\mathbb{G}_\varepsilon, \mathbf{d}_\varepsilon, \mathscr{L}^n) \Rightarrow$ pass to the limit (carefully!) as $\varepsilon \to 0$.

Idea of the proof: GF heat $\Rightarrow$ GF Ent

Assume $(u_t)_{t>0}$ solves $\partial_t u_t = \Delta_{\mathbb{G}} u_t$ with $u_0 \in L^1(\mathbb{G})$, $u_0 \mathscr{L}^n \in \mathrm{Dom}(\mathrm{Ent})$
$\Rightarrow u_t(x) = u_0 \star \mathbf{h}_t(x) = \int_{\mathbb{G}} \mathbf{h}_t(y^{-1}x) u_0(y)\, dy$, where $(\partial_t - \Delta_{\mathbb{G}})\mathbf{h}_t = \delta_0$
$\Rightarrow$ by Gaussian estimates on $\mathbf{h}_t$ (Varopoulos, Saloff-Coste, Coulhon)

$$\frac{d}{dt}\mathrm{Ent}(\mu_t) \overset{(\mathrm{HE})}{=} \int_{\mathbb{G}} (1 + \log u_t)\, \Delta_{\mathbb{G}} u_t \, dx = -\int_{\mathbb{G}} \frac{\|\nabla_{\mathbb{G}} u_t\|^2_{\mathbb{G}}}{u_t}\, d\mu_t.$$

Since $\partial_t u_t + \mathrm{div}(v_t u_t) = 0$ with $v_t = -\frac{\nabla_{\mathbb{G}} u_t}{u_t}$, the velocity of $(\mu_t)_{t>0}$ satisfies

$$|\dot{\mu}_t|^2 \le \int_{\mathbb{G}} \|v_t\|^2_{\mathbb{G}}\, dx = \int_{\mathbb{G}} \frac{\|\nabla_{\mathbb{G}} u_t\|^2_{\mathbb{G}}}{u_t}\, d\mu_t \quad \text{for a.e. } t > 0.$$

By the (metric) chain rule

$$\left|\frac{d}{dt}\mathrm{Ent}(\mu_t)\right| \le |D^-\mathrm{Ent}|(\mu_t) \cdot |\dot{\mu}_t| \qquad \text{for a.e. } t > 0.$$

We conclude by proving that

$$|D^-\mathrm{Ent}|^2(\mu_t) = \int_{\mathbb{G}} \frac{\|\nabla_{\mathbb{G}} u_t\|^2_{\mathbb{G}}}{u_t}\, d\mu_t \quad \leftarrow \text{Fisher information!}$$

($\le$ is always true, $\ge$ needs Riemannian approximation of $\mathbb{G}$)

Idea of the proof: GF Ent $\Rightarrow$ GF heat

Assume $(\mu_t)_{t>0}$ is a GF of Ent with $u_0 \in L^1(\mathbb{G})$, $u_0\mathscr{L}^n \in \mathrm{Dom}(\mathrm{Ent})$

$\Rightarrow \mu_t = u_t\mathscr{L}^n$ since $\mathrm{Ent}(\mu) < +\infty \iff \mu \ll \mathscr{L}^n$ by definition

$\Rightarrow (u_t)_{t>0}$ satisfies $\partial_t u_t + \mathrm{div}(v_t u_t) = 0$ by (CE) and for a.e. $t > 0$

$$-\frac{d}{dt}\mathrm{Ent}(\mu_t) \geq \frac{1}{2}|\dot{\mu}_t|^2 + \frac{1}{2}|D^-\mathrm{Ent}(\mu_t)|^2 \geq \|v_t\|_{L^2(\mu_t)} \cdot \left\| \frac{\nabla u_t}{u_t} \right\|_{L^2(\mu_t)}$$

Smooth in time and space (μ_t, v_t) and get $(\tilde{\mu}_t^\varepsilon, \tilde{v}_t^\varepsilon)$ (using group structure!)

$\Rightarrow (\tilde{\mu}_t^\varepsilon)_{t>0}$ is <u>not</u> a GF, but still $\partial_t \tilde{u}_t^\varepsilon + \mathrm{div}(\tilde{v}_t^\varepsilon \tilde{u}_t^\varepsilon) = 0$, hence

$$\int_{\mathbb{R}}\int_{\mathbb{G}} \langle \nabla_{\mathbb{G}}\phi_t, \tilde{v}_t^\varepsilon \rangle_{\mathbb{G}} \tilde{u}_t^\varepsilon \, dxdt = -\int_{\mathbb{R}}\int_{\mathbb{G}} \partial_t \phi_t \, \tilde{u}_t^\varepsilon \, dxdt = \int_{\mathbb{R}}\int_{\mathbb{G}} \phi_t \partial_t \tilde{u}_t^\varepsilon \, dxdt$$

$\Rightarrow$ we test $\phi_t = (1 + \log \tilde{u}_t^\varepsilon)$ and get

$$\frac{d}{dt}\mathrm{Ent}(\tilde{\mu}_t^\varepsilon) = \int_{\mathbb{G}} (1 + \log \tilde{u}_t^\varepsilon)\, \partial_t \tilde{u}_t^\varepsilon \, dx = \int_{\mathbb{G}} \left\langle \frac{\nabla_{\mathbb{G}} \tilde{u}_t^\varepsilon}{\tilde{u}_t^\varepsilon}, \tilde{v}_t^\varepsilon \right\rangle d\tilde{\mu}_t^\varepsilon.$$

$\Rightarrow$ we pass to the limit as $\varepsilon \to 0$ and conclude by Cauchy-Schwarz

Futurama

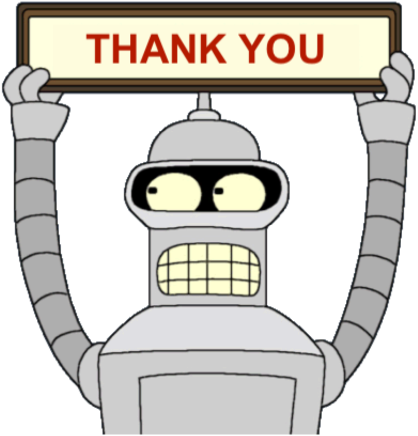
- Does the correspondence hold in other sub-Riemannian manifolds?

  → Nihilmanifolds $M = \mathbb{G}/H$ with $H \leq \mathbb{G}$ closed unimodular not normal
  Good news: Gaussian estimates on heat kernel in [Maheux, 1998]
  Bad news: NO Riemannian approximation! NO convolution!

  → Grushin spaces (plane: $M = \mathbb{R}^2$ with $X = \partial_x$, $Y = x\partial_y$)
  Good news: Riemannian approximation ($X = \partial_x$, $Y_\varepsilon = \sqrt{|x|^2 + \varepsilon}\partial_y$)
  To understand: regularization?

- Can we find a generalized CD condition for Carnot groups?

  → For step 2 YES! [Baudoin-Garofalo, 2014]
  Link with Ent-inequalities in [Balogh-Kristaly-Sipos] & [Barilari-Rizzi]?


THANK YOU