Exercises for week 1

Giorgio Stefani

Exercise 1 (4 points + 2 bonus points)**.**

(i) [2 points] Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ be defined by

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,\ y \text{ and } 0 \text{ are collinear,} \\ |x| + |y| & \text{otherwise,} \end{cases}$$

for all $x, y \in \mathbb{R}^2$. Prove that $(\mathbb{R}^2, d)$ is a metric space. Is $(\mathbb{R}^2, d)$ complete? Motivate your answer.

(ii) [2 points + 2 bonus points] Let $d\colon \mathbb{N} \times \mathbb{N} \to [0, +\infty)$, where $\mathbb{N} = \{1, 2, 3, \dots\}$, be defined by

$$d(m,n) = \left| \frac{1}{m} - \frac{1}{n} \right| \quad \text{for all } m, n \in \mathbb{N}.$$

(a) [2 points] Prove that $(\mathbb{N}, d)$ is a metric space and that is not complete.

(b) [2 bonus points] Find a completion of $(\mathbb{N}, d)$.

Exercise 2 (4 points)**.** Let $a, b \in \mathbb{R}$, with $a < b$, and let $X = C([a,b];\mathbb{R})$ be the set of continuous functions $f\colon [a,b] \to \mathbb{R}$.

(i) [2 points] Let $d\colon X \times X \to [0, +\infty)$ be defined as

$$d(f,g) = \int_a^b |f(x) - g(x)| \,\mathrm{d}x$$

for all $f, g \in X$. Prove that (X, d) is a metric space.

(ii) [2 points] Is (X, d) complete? Motivate your answer.

Exercise 3 (4 points)**.** Let X be a non-empty set and let (Y, d) be a metric space. Assume that $\varphi\colon X \to Y$ is injective. Let $d_\varphi\colon X \times X \to \mathbb{R}$ be defined as

$$d_\varphi(x, x') = d(\varphi(x), \varphi(x')) \quad \text{for all } x, x' \in X.$$

(i) [1 point] Prove that (X, d_φ) is a metric space.

(ii) [3 points] Prove that (X, d_φ) is complete if and only if $(\varphi(X), d)$ is complete.

Exercise 4 (4 points)**.** Let $A, B \subset \mathbb{R}^n$ be two non-empty sets. Let us define

$$\operatorname{dist}(A, B) = \inf \big\{ |x - y| : x \in A,\ y \in B \big\}.$$

(i) [2 points] Prove that if A is compact and B is closed and $A \cap B = \varnothing$, then $d(A, B) > 0$.

(ii) [2 points] Is it true that $d(A, B) > 0$ for all $A, B \subset \mathbb{R}^n$ non-empty, closed and such that $A \cap B = \varnothing$? If yes, then prove the statement. If no, then give a counterexample.

Exercises for week 2

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Exercise 5 (4 points)**.** Let (X,d) be a compact metric space. Assume that $T\colon X\to X$ satisfies

$$d(T(x),T(y))<d(x,y)\quad \text{for all } x,y\in X,\ x\neq y.$$

(i) [1 point] Prove that, if $x,y\in X$ are fix points of the map T, then $x=y$.
(ii) [1 point] Prove that the map $f\colon X\to\mathbb{R}$, defined as $f(x)=d(T(x),x)$ for all $x\in X$, is continuous.
(iii) [2 points] Prove that the map T has a unique fix point $\bar{x}\in X$.

Exercise 6 (4 points)**.** Let (X,d) be a metric space and let $A\subset X$. Let $L>0$ and let $f\colon A\to\mathbb{R}$ be an L-Lipschitz function on A, that is, such that

$$|f(x)-f(y)|\le L\,d(x,y)\quad \text{for all } x,y\in A.$$

Prove the following two statements.

(i) [2 points] There exists an L-Lipschitz function $F\colon X\to\mathbb{R}$ on X such that $F\big|_A=f$.
(ii) [2 points] There exist two L-Lipschitz functions $m,M\colon X\to\mathbb{R}$ on X such that $m\big|_A=M\big|_A=f$ and with the following property: if $g\colon X\to\mathbb{R}$ is an L-Lipschitz function on X such that $g\big|_A=f$, then $m\le g\le M$ on X.

Exercise 7 (4 points)**.** Let (X,d) be a metric space. Recall that the *diameter* of a non-empty set $A\subset X$ is defined as $\operatorname{diam}(A)=\sup\{d(x,y):x,y\in A\}$.

(i) [3 points] Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X such that:
   (a) C_n is non-empty and closed for all $n\in\mathbb{N}$;
   (b) $C_{n+1}\subset C_n$ for all $n\in\mathbb{N}$;
   (c) $\lim_{n\to+\infty}\operatorname{diam}(C_n)=0$.

   Prove that, if (X,d) is complete, then there exists a unique $x\in X$ such that $\bigcap_{n\in\mathbb{N}} C_n=\{x\}$.
(ii) [1 point] Find a metric space (X,d) and a sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of X satisfying properties (a), (b) and (c) of the previous point and such that $\bigcap_{n\in\mathbb{N}} C_n=\varnothing$.

Exercise 8 (4 points)**.** Let $p\in[1,+\infty]$ and let $(\ell^p(\mathbb{N}),\|\cdot\|_{\ell^p(\mathbb{N})})$ be the Banach space of p-summable real sequences. Prove that $(\ell^p(\mathbb{N}),\|\cdot\|_{\ell^p(\mathbb{N})})$ is a Hilbert space if [2 points] and only if [2 points] $p=2$.

Bonus (4 points)**.** Let us define

$$\mathcal{S}(\mathbb{R})=\left\{f\in C^\infty(\mathbb{R}):\sup_{x\in\mathbb{R}}|x|^m\,|f^{(n)}(x)|<+\infty \text{ whenever } m,n\in\mathbb{N}_0\right\},$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $C^\infty(\mathbb{R})$ is the set of infinitely differentiable functions and $f^{(n)}$ denotes the n-th derivative of f. Prove that, if $f\in\mathcal{S}(\mathbb{R})$, then $f^{(n)}$ is uniformly continuous on $\mathbb{R}$ for all $n\in\mathbb{N}_0$.

Exercises for week 3

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Exercise 9 (4 points)**.** Let $(X, \mathscr{A}, \mu)$ be a measurable space. A point $x \in X$ is called an *atom* of μ if $\{x\} \in \mathscr{A}$ and $\mu(\{x\}) > 0$. From now on, we assume $\mu(X) < +\infty$.

(i) [2 points] Prove that the set

$$A_n = \left\{ x \in X : x \text{ is an atom of } \mu \text{ with } \mu(\{x\}) \geq \frac{\mu(X)}{n} \right\}$$

is finite for all $n \in \mathbb{N}$.

(ii) [2 points] Prove that the set $A = \{x \in X : x \text{ is an atom of } \mu\}$ is at most countable.

Exercise 10 (4 points)**.** Let X be a non-empty set and let $\mathscr{Q} \subset \mathscr{P}(X)$. We define

$$\sigma(\mathscr{Q}) = \bigcap \left\{ \mathscr{A} \text{is a } \sigma\text{-algebra of } X : \mathscr{A} \supset \mathscr{Q} \right\}.$$

(i) [2 points] Prove that $\sigma(\mathscr{Q})$ is a σ-algebra of X and that $\sigma(\mathscr{Q}) \supset \mathscr{Q}$.

(ii) [2 points] Prove that $\sigma(\mathscr{Q})$ is the smallest σ-algebra of X containing $\mathscr{Q}$, that is, if $\mathscr{A}$ is a σ-algebra such that $\mathscr{A} \supset \mathscr{Q}$, then $\mathscr{A} \supset \sigma(\mathscr{Q})$.

Exercise 11 (6 points)**.** Let $n \in \mathbb{N}$ be fixed and let $\mu\colon \mathscr{P}(X) \to [0, +\infty]$ be an outer measure on $\mathbb{R}^n$. Assume that there exist two constants $C, \varepsilon > 0$ such that

$$\mu\left(B_r(x)\right) \leq C r^{n+\varepsilon}$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ is the open Euclidean ball.

(i) We assume $n = 1$ first.

(a) [1 point] Prove that $\mu((a, b)) \leq \tilde{C}(b - a)^{1+\varepsilon}$ for all $a, b \in \mathbb{R}$, with $a < b$, where $\tilde{C} > 0$ is a constant depending uniquely on C and ε.

(b) [1 point] Prove that $\mu((a, b)) \leq \dfrac{\tilde{C}(b - a)^{1+\varepsilon}}{k^\varepsilon}$ for all $k \in \mathbb{N}$ and for all $a, b \in \mathbb{R}$, with $a < b$.

(c) [1 point] Prove that $\mu \equiv 0$, i.e. $\mu(E) = 0$ for all $E \in \mathscr{P}(\mathbb{R})$.

(ii) Now we assume $n \geq 2$.

(a) [2 points] Prove that $\mu\left(Q_r(x)\right) \leq \tilde{C} r^{n+\varepsilon}$ for all $x \in \mathbb{R}^n$ and $r > 0$, where $Q_r(x) = \left(x_1 - \frac{r}{2}, x_1 + \frac{r}{2}\right) \times \cdots \times \left(x_n - \frac{r}{2}, x_n + \frac{r}{2}\right)$ is the Euclidean box centered at $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with side length $r > 0$, and $\tilde{C} > 0$ is a constant depending uniquely on C and ε.

(b) [1 point] Prove that $\mu \equiv 0$, i.e. $\mu(E) = 0$ for all $E \in \mathscr{P}(\mathbb{R}^n)$.

Bonus (4 points)**.** Let X and Y be two non-empty sets and let $\mathscr{B} \subset \mathscr{P}(Y)$ be a σ-algebra of Y. Let $f\colon X \to Y$ be a function and define

$$\sigma(f) = \{f^{-1}(S) : S \in \mathscr{B}\} \subset \mathscr{P}(X).$$

(i) [2 points] Prove that $\sigma(f)$ is a σ-algebra of X.

(ii) [2 points] Let $\mathscr{A} \subset \mathscr{P}(X)$ be a σ-algebra of X. Prove that $f\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ is measurable, that is, $f^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}$, if and only if $\mathscr{A} \supset \sigma(f)$.

Exercises for week 4

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Exercise 12 (4 points)**.** Let (X,d) be a metric space and let $C \subset X$ be non-empty and closed.

(1) [1 point] Prove that $A_k = \{x \in X : \operatorname{dist}(x,C) < \frac{1}{k}\}$ is open for all $k \in \mathbb{N}$.
(2) [2 points] Prove that $C = \cap_{k\in\mathbb{N}} A_k$.
(3) [1 point] Prove that an open set $A \subset X$ is a countable union of closed sets.

Exercise 13 (5 points)**.** Let $n \in \mathbb{N}$ and let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function with Lipschitz constant $L \in [0,+\infty)$. Let $s \geq 0$ and let $\mathscr{H}^s\colon \mathscr{P}(\mathbb{R}^n) \to [0,+\infty]$ be the s-dimensional Hausdorff outer measure on $\mathbb{R}^n$.

(i) [2 points] Prove that $\operatorname{diam}(f(E)) \leq L \operatorname{diam}(E)$ for all $E \subset \mathbb{R}^n$.
(ii) [2 points] Prove that $\mathscr{H}^s(f(E)) \leq L^s\, \mathscr{H}^s(E)$ for all $E \subset \mathbb{R}^n$.
(iii) [1 point] Find a constant $L \in (0,+\infty)$ and an L-Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathscr{H}^s(f(E)) = L^s \mathscr{H}^s(E)$ for all $E \subset \mathbb{R}^n$.

Exercise 14 (7 points)**.** Let (X,d) be a metric space. Let $\mathscr{F} \subset \mathscr{P}(X)$ be a family of subsets of X such that $\varnothing \in \mathscr{F}$. Let $\tau\colon \mathscr{F} \to [0,+\infty]$ be a function such that $\tau(\varnothing) = 0$. Given $\delta > 0$, we define $\mu^\tau_\delta\colon \mathscr{P}(X) \to [0,+\infty]$ as

$$\mu^\tau_\delta(A) = \inf\left\{\sum_{k=1}^{+\infty} \tau(E_k) : (E_k)_{k\in\mathbb{N}} \subset \mathscr{F},\ \operatorname{diam}(E_k) \leq \delta\ \forall k \in \mathbb{N},\ A \subset \bigcup_{k=1}^{+\infty} E_k\right\}$$

for all $A \subset X$.

(i) [2 points] Prove that the limit $\mu^\tau(A) = \lim_{\delta\to 0^+} \mu^\tau_\delta(A) \in [0,+\infty]$ is well defined for all $A \subset X$.
(ii) [2 points] Prove that the function $\mu^\tau\colon \mathscr{P}(X) \to [0,+\infty]$ is an outer measure on X.
(iii) [2 points] Prove that the function $\mu^\tau\colon \mathscr{P}(X) \to [0,+\infty]$ is a metric outer measure on X, that is, if $\operatorname{dist}(A,B) > 0$, then $\mu^\tau(A \cup B) = \mu^\tau(A) + \mu^\tau(B)$ for all $A,B \subset X$.
(iv) [1 point] Prove that $\mu^\tau\colon \mathscr{P}(X) \to [0,+\infty]$ is a Borel outer measure on X.

Exercise 15 (7 points)**.** Let $\mathscr{L}^n\colon \mathscr{P}(\mathbb{R}^n) \to [0,+\infty]$ be the Lebesgue outer measure on $\mathbb{R}^n$.

(i) [1 point] Prove that $\mathscr{L}^1(\partial I) = 0$ for all open intervals $I \subset \mathbb{R}$.
(ii) [2 points] Prove that $\mathscr{L}^n(\partial I) = 0$ for all open boxes $I \subset \mathbb{R}^n$.
(iii) [3 points] Let $H = \{x \in \mathbb{R}^n : x_n = 0\}$. Prove that $\mathscr{L}^n(H) = 0$.
(iv) [1 point] Prove that $\mathscr{L}^n(A) > 0$ for all non-empty open sets $A \subset \mathbb{R}^n$.

Bonus (2 points)**.** Let X be a non-empty set and let $\mu\colon \mathscr{P}(X) \to [0,+\infty]$ be an outer measure. Prove that $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for all $A, B \in \operatorname{Meas}(X;\mu)$.

Exercises for week 5

Giorgio Stefani

Exercise 16 (4 points)**.** Let (X, d) be a metric space.

(i) [2 points] Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of non-empty compact subsets of X such that $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$. Prove that $\bigcap_{n\in\mathbb{N}} K_n \neq \varnothing$.

(ii) [2 points] Prove that $\operatorname{diam}(\mathring{A}) \leq \operatorname{diam}(A) = \operatorname{diam}(\overline{A})$ for all $A \subset X$. As usual, $\operatorname{diam}(\varnothing) = 0$.

Exercise 17 (6 points)**.** Let $\phi\colon [0, +\infty) \to [0, +\infty)$ be a continuous function such that

(a) $\phi(0) = 0$;
(b) $s \leq t \Rightarrow \phi(s) \leq \phi(t)$ for all $s, t \in [0, +\infty)$;
(c) $\phi(s + t) \leq \phi(s) + \phi(t)$ for all $s, t \in [0, +\infty)$.

We define

$$\phi(+\infty) = \lim_{t\to+\infty} \phi(t) \in [0, +\infty].$$

Let X be a non-empty set and let $\mu\colon \mathscr{P}(X) \to [0, +\infty]$ be an outer measure.

(i) [2 points] Find two continuous functions $\phi_1, \phi_2\colon [0, +\infty) \to [0, +\infty)$, satisfying points (a), (b) and (c) above, such that $\phi_1(+\infty) = +\infty$ and $\phi_2(+\infty) \in (0, +\infty)$.

(ii) [2 points] Prove that the function $\mu_\phi\colon \mathscr{P}(X) \to [0, +\infty]$ defined by setting $\mu_\phi(A) = \phi(\mu(A))$ for all $A \subset X$ is an outer measure on X.

(iii) [2 points] Assume that $\phi\colon [0, +\infty) \to [0, +\infty)$ is injective. Prove that $\operatorname{Meas}(X; \mu_\phi) \subset \operatorname{Meas}(X; \mu)$.

Exercise 18 (6 points)**.** Let $s \geq 0$ and let $\mathscr{H}^s\colon \mathscr{P}(\mathbb{R}^n) \to [0, +\infty]$ be the s-dimensional Hausdorff outer measure on $\mathbb{R}^n$.

(i) [2 points] Prove that, if $s > n$, then $\mathscr{H}^s(A) = 0$ for all $A \subset \mathbb{R}^n$.

(ii) [2 points] Prove that, if $\mathscr{H}^s(A) < +\infty$ for some $A \subset \mathbb{R}^n$, then $\mathscr{H}^t(A) = 0$ for all $t \in (s, +\infty)$.

(iii) [2 points] Prove that, if $\mathscr{H}^s(A) > 0$ for some $A \subset \mathbb{R}^n$, then $\mathscr{H}^t(A) = +\infty$ for all $t \in [0, s)$.

Bonus (4 points)**.** Set $\aleph_0 = \operatorname{Card}(\mathbb{N})$ as usual. Let X be a set with $\operatorname{Card}(X) > \aleph_0$.

(i) [2 points] Define

$$\mathscr{A} = \Big\{ A \subset X : \text{either } \operatorname{Card}(A) \leq \aleph_0 \text{ or } \operatorname{Card}(A^c) \leq \aleph_0 \Big\}.$$

Prove that $\mathscr{A}$ is a σ-algebra.

(ii) [2 points] Define $\mu\colon \mathscr{A} \to [0, +\infty]$ by setting

$$\mu(A) = \begin{cases} 0 & \text{if } \operatorname{Card}(A) \leq \aleph_0, \\ 1 & \text{if } \operatorname{Card}(A^c) \leq \aleph_0, \end{cases}$$

for all $A \in \mathscr{A}$. Prove that $(X, \mathscr{A}, \mu)$ is a measurable space.

Exercises for week 6

Giorgio Stefani

Exercise 19 (4 points)**.** Let X and Y be two sets and let $f\colon X \to Y$ be a function. Let I be an arbitrary set of indexes and assume $A_i \subset X$ and $B_i \subset Y$ for all $i \in I$. Prove the following statements.

(i) [1 point] $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$.

(ii) [1 point] $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$.

(iii) [1 point] $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$.

(iv) [1 point] $f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f(A_i)$, with possible strict inclusion (example required).

Exercise 20 (10 points)**.** Let $s \ge 0$ and let $\mathscr{H}^s\colon \mathscr{P}(\mathbb{R}^n) \to [0,+\infty]$ be the s-dimensional Hausdorff outer measure on $\mathbb{R}^n$.

(i) [2 points] Prove that $\mathscr{H}^0$ coincides with the counting outer measure. Precisely, $\mathscr{H}^0(A) = \mathsf{c}(A)$ for all $A \subset \mathbb{R}^n$, where $\mathsf{c}\colon \mathscr{P}(\mathbb{R}^n) \to [0,+\infty]$ is defined as

$$\mathsf{c}(A) = \begin{cases} \mathrm{Card}(A) & \text{if } \mathrm{Card}(A) < +\infty, \\ +\infty & \text{otherwise} \end{cases}$$

for all $A \subset \mathbb{R}^n$. *Hint*: start by proving that $\mathscr{H}^0(\{x\}) = 1$ for all $x \in \mathbb{R}^n$.

(ii) [2 points] Prove that, if $\mathscr{H}^s_\delta(A) = 0$ for some $A \subset \mathbb{R}^n$ and $\delta \in (0,+\infty)$, then $\mathscr{H}^s(A) = 0$.

(iii) [2 points] Let $\varnothing \ne A \subset \mathbb{R}$ be such that $\mathrm{diam}(A) < +\infty$. Prove that $-\infty < \inf A \le \sup A < +\infty$ and that $\mathrm{diam}(A) = \sup A - \inf A$.

(iv) [2 points] Prove that

$$(\star) \qquad \mathscr{L}^1(A) = \inf\left\{\sum_{k=1}^{+\infty} \mathrm{diam}(E_k) : A \subset \bigcup_{k=1}^{+\infty} E_k\right\}$$

for all $A \subset \mathbb{R}$, where $\mathscr{L}^1\colon \mathscr{P}(\mathbb{R}) \to [0,+\infty]$ is the Lebesgue outer measure on $\mathbb{R}$. Deduce that $\mathscr{L}^1(A) \le \mathscr{H}^1(A)$ for all $A \subset \mathbb{R}$ (you may need to know that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$).

(v) [2 points] Let $\delta > 0$. Exploiting $(\star)$ and the family $(I^\delta_k)_{k\in\mathbb{Z}}$, where $I^\delta_k = [k\delta, (k+1)\delta]$ for all $k \in \mathbb{Z}$, prove that $\mathscr{L}^1(A) = \mathscr{H}^1(A)$ for all $A \subset \mathbb{R}$.

Exercise 21 (4 points)**.** Let $\dim_{\mathscr{H}}(A) \in [0,n]$ be the Hausdorff dimension of the set $A \subset \mathbb{R}^n$.

(i) [2 points] Prove that, if $A \subset \mathbb{R}^n$ is countable, then $\dim_{\mathscr{H}}(A) = 0$.

(ii) [2 points] Prove that, if $A, B \subset \mathbb{R}^n$, then $\dim_{\mathscr{H}}(A \cup B) = \max\{\dim_{\mathscr{H}}(A), \dim_{\mathscr{H}}(B)\}$.

Exercise 22 (4 points)**.** Let $\mathscr{L}^n\colon \mathscr{P}(\mathbb{R}^n) \to [0,+\infty]$ be the Lebsegue outer measure on $\mathbb{R}^n$.

(i) [2 points] Let $E \subset \mathbb{R}^n$. Prove that $E \in \mathrm{Meas}(\mathbb{R}^n; \mathscr{L}^n)$ if and only if $\mathbf{1}_E\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{L}^n$-measurable, where

$$\mathbf{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

for all $x \in \mathbb{R}^n$.

(ii) [2 points] Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \pi \min\{x^2, 3\} & x \in \mathbb{Q} \\ -\sqrt{2}\,x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is $\mathscr{L}^1$-measurable.

Exercises for week 7

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Exercise 23 (4 points)**.** Let (X, d) be a metric space. Let $A, B \subset X$ and let $f\colon X \to \mathbb{R}$ be a function.

(i) [3 points] Assume that $\operatorname{dist}(A, B) > 0$. Prove that, if the functions $f\colon A \to \mathbb{R}$ and $f\colon B \to \mathbb{R}$ are continuous, then also $f\colon A \cup B \to \mathbb{R}$ is continuous.

(ii) [1 point] Is the implication in (i) still true if we only assume that $A \cap B = \varnothing$? If yes, then prove it. If no, then give an explicit counterexample.

Exercise 24 (3 points)**.** Let X be a non-empty set and let $\mu\colon \mathscr{P}(X) \to [0, +\infty]$ be an outer measure on X. Assume that $f_k\colon X \to \mathbb{R}$ is μ-measurable for all $k \in \mathbb{N}$. Prove that the sets

$$\left\{x \in X : \exists \lim_{k\to+\infty} f_k(x) \in \mathbb{R}\right\}, \quad \left\{x \in X : \lim_{k\to+\infty} f_k(x) = +\infty\right\}, \quad \left\{x \in X : \lim_{k\to+\infty} f_k(x) = -\infty\right\},$$

are μ-measurable.

Exercise 25 (4 points)**.** Let $\mu\colon \mathscr{P}(\mathbb{R}^n) \to [0, +\infty]$ be Radon outer measure on $\mathbb{R}^n$. Let $f_k\colon \mathbb{R}^n \to \mathbb{R}$ be a μ-measurable function for all $k \in \mathbb{N}$. Assume that

(a) $E \in \operatorname{Meas}(\mathbb{R}^n; \mu)$ with $\mu(E) < +\infty$;

(b) for each $x \in E$ there is a constant $M_x \in \mathbb{R}$ such that $|f_k(x)| \le M_x$ for all $k \in \mathbb{N}$.

Prove that, for each $\varepsilon > 0$, there exist a set $F_\varepsilon \subset E$ and a constant $M_\varepsilon \in \mathbb{R}$ such that

(i) F_ε is closed and $\mu(E \setminus F_\varepsilon) < \varepsilon$;

(ii) $|f_k(x)| \le M_\varepsilon$ for all $x \in F_\varepsilon$ and all $k \in \mathbb{N}$.

Exercise 26 (4 points)**.** Let $\mu\colon \mathscr{P}(\mathbb{R}^n) \to [0, +\infty]$ be a Borel outer measure on $\mathbb{R}^n$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function. Assume that, if $\varepsilon > 0$, then there exists a closed set $C \subset \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus C) < \varepsilon$ and $f\colon C \to \mathbb{R}$ is continuous. Prove that $f\colon \mathbb{R}^n \to \mathbb{R}$ is μ-measurable.

Bonus (4 points)**.** Let $\mathscr{L}^n\colon \mathscr{P}(\mathbb{R}^n) \to [0, +\infty]$ be the Lebesgue outer measure. Find an $\mathscr{L}^2$-measurable set $E \subset \mathbb{R}^2$ and a smooth and Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(E) \subset \mathbb{R}$ is not $\mathscr{L}^1$-measurable.

Exercises for week 8

Giorgio Stefani

Exercise 27 (6 points + 2 bonus points)**.** Let $f\colon [0,1]\to\mathbb{R}$ be a bounded function. We let

$$\omega(f,I)=\sup\{f(x)-f(y):x,y\in I\}$$

be the oscillation of the function f on the interval $I\subset[0,1]$. We also let

$$\omega(f,x)=\lim_{r\to0^+}\omega(f,I_r(x))$$

be the local oscillation of the function f at $x\in[0,1]$, where $I_r(x)=\{y\in[0,1]:|y-x|<r\}$.

(i) [1 point] Prove that $\omega(f,x)=\inf_{r>0}\omega(f,I_r(x))$ for all $x\in[0,1]$.

(ii) [1 point] Prove that $x\mapsto\omega(f,x)$ is upper semicontinuous on $[0,1]$, that is,

$$\limsup_{y\to x}\omega(f,y)\le\omega(f,x)\quad\text{for all } x\in[0,1].$$

(iii) [2 points] Prove that f is continuous at $x\in[0,1]$ if and only if $\omega(f,x)=0$.

(iv) [2 points] Assume that $\omega(f,x)<\varepsilon$ for all $x\in[0,1]$ for some $\varepsilon>0$. Prove that there exists $\delta>0$ such that $\omega(f,I)<\varepsilon$ for all intervals $I\subset[0,1]$ such that $\mathscr{L}^1(I)<\delta$.

(v) [2 bonus points] Let $D(f)=\{x\in[0,1]:\omega(f,x)>0\}$. Prove that, if $\mathscr{L}^1(D(f))=0$, then f is $\mathscr{L}^1$-measurable on $[0,1]$.

Exercise 28 (4 points)**.** Let $f\colon X\to[0,+\infty]$ be a μ-measurable function.

(i) [2 points] Prove that $\mu\big(\{x\in X:f(x)>t\}\big)\le\frac{1}{t}\int_X f\,d\mu$ for all $t>0$.

(ii) [2 points] Prove that, if $\int_X f\,d\mu<+\infty$, then $f(x)<+\infty$ for μ-a.e. $x\in X$.

Exercise 29 (4 points)**.** Let $f\in\mathbb{R}\to\mathbb{R}$ be a $\mathscr{L}^1$-measurable such that $\int_{\mathbb{R}}|f|\,d\mathscr{L}^1<+\infty$. Let $c\in\mathbb{R}$. Prove that the function $F_c\colon\mathbb{R}\to\mathbb{R}$ defined by $F_c(t)=\int_{[c,t]^*}f\,d\mathscr{L}^1$ for all $t\in\mathbb{R}$, where $[c,t]^*=\{x\in\mathbb{R}:c\wedge t\le x\le c\vee t\}$, is well defined and continuous on $\mathbb{R}$.

Exercise 30 (4 points)**.** Let $f_k\colon X\to[0,+\infty]$ be a non-negative μ-measurable function for all $k\in\mathbb{N}$. Assume that there exists a function $f\colon X\to[0,+\infty]$ such that

(a) $\lim_{k\to+\infty}f_k(x)=f(x)$ for μ-a.e. $x\in X$;

(b) $f_k(x)\le f(x)$ for μ-a.e. $x\in X$ and all $k\in\mathbb{N}$.

Prove that $\lim_{k\to+\infty}\int_X f_k\,d\mu=\int_X f\,d\mu$.

Bonus (4 points)**.** Let $f\colon X\to[0,+\infty]$ be a non-negative μ-measurable function. Prove that the function $\nu\colon\mathrm{Meas}(X;\mu)\to[0,+\infty]$, defined as

$$\nu(A)=\int_A f\,d\mu$$

for all $A\in\mathrm{Meas}(X;\mu)$, is a measure.

Exercises for week 9

Giorgio Stefani

Exercise 31 (6 points).

(i) [2 points] Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be two continuous functions. Prove that, if $f = g$ $\mathscr{L}^n$-a.e. on $\mathbb{R}^n$, then $f = g$ everywhere on $\mathbb{R}^n$.

(ii) [2 points] Let $f, g \in \mathcal{L}^1(X, \mu)$. Prove that $f = g$ μ-a.e. on X if and only if $\int_E f\,d\mu = \int_E g\,d\mu$ for all $E \in \mathrm{Meas}(X;\mu)$.

(iii) [2 points] Set $\aleph_0 = \mathrm{Card}(\mathbb{N})$ as usual. Let X be a set with $\mathrm{Card}(X) > \aleph_0$. Consider

$$\mathscr{A} = \left\{ A \subset X : \text{either } \mathrm{Card}(A) \le \aleph_0 \text{ or } \mathrm{Card}(A^c) \le \aleph_0 \right\}$$

and define $\mu\colon \mathscr{A} \to [0, +\infty]$ by setting

$$\mu(A) = \begin{cases} 0 & \text{if } \mathrm{Card}(A) \le \aleph_0 \\ +\infty & \text{if } \mathrm{Card}(A^c) \le \aleph_0 \end{cases}$$

for all $A \in \mathscr{A}$. Prove that $(X, \mathscr{A}, \mu)$ is a measurable space [1 point] and find two μ-measurable functions $f, g\colon X \to [0, +\infty]$ such that $\int_A f\,d\mu = \int_A g\,d\mu$ for all $A \in \mathscr{A}$ but $f(x) \ne g(x)$ for all $x \in X$ [1 point].

Exercise 32 (4 points). Let $f_k\colon X \to [-\infty, +\infty]$ be a μ-measurable function for all $k \in \mathbb{N}$ and let $g \in \mathcal{L}^1(X, \mu)$. Prove that, if $f_k \le g$ μ-a.e. on X for all $k \in \mathbb{N}$, then

$$\limsup_{k \to +\infty} \int_X f_k\,d\mu \le \int_X \limsup_{k \to +\infty} f_k\,d\mu.$$

Exercise 33 (4 points). Let $f_k \in \mathcal{L}^1(X, \mu)$ be such that $f_k \ge 0$ μ-a.e. on X for all $k \in \mathbb{N}$. Assume that

$$\lim_{k \to +\infty} f_k = f \ \mu\text{-a.e. on } X \quad \text{and} \quad \lim_{k \to +\infty} \int_X f_k\,d\mu = \int_X f\,d\mu < +\infty.$$

Prove that $\lim_{k \to +\infty} \int_X |f_k - f|\,d\mu = 0$.

Exercise 34 (4 points). Let $f\colon X \to \mathbb{R}$ be an μ-measurable function such that $f \ge 0$ μ-a.e. on X. Assume that there exists $\alpha \in [0, +\infty)$ such that $\int_X f(x)^k\,d\mu(x) = \alpha$ for all $k \in \mathbb{N}$. Prove that $f = \mathbf{1}_E$ μ-a.e. on X for some $E \in \mathrm{Meas}(X;\mu)$ with $\mu(E) = \alpha$.

Bonus (4 points). Let X be an non-empty set and let $\mathscr{R} \subset \mathscr{P}(X)$ be a semiring on X. Let $\mu\colon \mathscr{R} \to [0, +\infty]$ be a measure on $\mathscr{R}$.

(1) [2 points] Let $A, B_1, \dots, B_m \in \mathscr{R}$. Prove that there exist disjoint sets $C_1, \dots, C_n \in \mathscr{R}$ such that $A \setminus \bigcup_{i=1}^m B_i = \bigsqcup_{j=1}^n C_j$.

(2) [2 points] Let $A_1, \dots, A_n \in \mathscr{R}$ be disjoint and assume that $\bigsqcup_{k=1}^n A_k \subset A$ for some $A \in \mathscr{R}$. Prove that $\sum_{k=1}^n \mu(A_k) \le \mu(A)$.

Exercises for week 10

Giorgio Stefani

Exercise 35 (6 points)**.** Let $f\colon X \to [0,+\infty]$ be a non-negative μ-measurable function and let $(A_k)_{k\in\mathbb{N}}$ be a sequence of μ-measurable sets.

(i) [1 point] Define $\gamma(x) = \sum\limits_{k=1}^{+\infty} \mathbf{1}_{A_k}(x)$ for all $x \in X$. Prove that $\gamma\colon X \to [0,+\infty]$ is μ-measurable.

(ii) [2 points] Let $A = \bigcup\limits_{k=1}^{+\infty} A_k$. Prove that $\int_A f\,d\mu \le \int_A f\gamma\,d\mu = \sum\limits_{k=1}^{+\infty} \int_{A_k} f\,d\mu$.

(iii) [2 points] Assume $M = \sup\limits_{x\in X} \gamma(x) < +\infty$. Prove that $\sum\limits_{k=1}^{+\infty} \int_{A_k} f\,d\mu \le M \int_A f\,d\mu$.

(iv) [1 point] Prove that $\sum\limits_{k=1}^{+\infty} \mu(A_k) \le M\mu\left(\bigcup\limits_{k=1}^{+\infty} A_k\right)$.

Exercise 36 (5 points)**.** Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{L}^1(X,\mu)$ be a sequence of μ-integrable functions such that $\sum_{k=1}^{+\infty} \int_X |f_k|\,d\mu < +\infty$.

(i) [2 points] Prove that $f(x) = \sum\limits_{k=1}^{+\infty} f_k(x) = \lim\limits_{m\to+\infty} \sum\limits_{k=1}^{m} f_k(x)$ exists and is finite for μ-a.e. $x \in X$.

(ii) [2 points] Prove that $f \in \mathcal{L}^1(X,\mu)$ and that $\lim\limits_{m\to+\infty} \displaystyle\int_X \left| f - \sum_{k=1}^{m} f_k \right| d\mu = 0$.

(iii) [1 point] Prove that $\int_X f\,d\mu = \sum\limits_{k=1}^{+\infty} \int_X f_k\,d\mu$.

Exercise 37 (4 points)**.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) = ye^{-(1+x^2)y^2}$ for $x, y \in \mathbb{R}$. Apply the Fubini–Tonelli's Theorem to the integral

$$I = \int_{\mathbb{R}\times[0,+\infty)} f(x,y)\,d(\mathscr{L}^1 \otimes \mathscr{L}^1)(x,y)$$

to show that $\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$.

Exercise 38 (4 points)**.** Let $f\colon X \to [0,+\infty]$ be a non-negative μ-measurable function. Define the set $G_f = \{(x,t) \in X \times \mathbb{R} : 0 \le t < f(x)\} \subset X \times \mathbb{R}$.

(i) [2 points] Prove that $G_f \in \mathrm{Meas}(X \times \mathbb{R}; \mu \otimes \mathscr{L}^1)$.

(ii) [2 points] Prove that $(\mu \otimes \mathscr{L}^1)(G_f) = \int_X f\,d\mu \in [0,+\infty]$.

Bonus (4 points)**.** Let $f \in \mathcal{L}^1(X,\mu)$ be a μ-integrable function. Prove that there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of simple functions $\varphi_k\colon X \to \mathbb{R}$, $k \in \mathbb{N}$, such that $\lim\limits_{k\to+\infty} \int_X |f - \varphi_k|\,d\mu = 0$.

Exercises for week 11

Giorgio Stefani

Exercise 39 (4 points). Let $(X, \mathscr{A}, \mu)$ be a measurable space.

(i) [2 points] Let $p, q, r \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Prove that, if $f \in \mathcal{L}^p(X, \mu)$ and $f \in \mathcal{L}^q(X, \mu)$, then $fg \in \mathcal{L}^r(X, \mu)$ with $\|fg\|_{\mathcal{L}^r} \le \|f\|_{\mathcal{L}^p} \|g\|_{\mathcal{L}^q}$.

(ii) [2 points] Let $m \in \mathbb{N}$ and let $p_1, \dots, p_m, r \in [1, +\infty]$ be such that $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{r}$. Prove that, if $f_k \in \mathcal{L}^{p_k}(X, \mu)$ for all $k = 1, \dots, m$, then $\prod_{k=1}^m f_k \in \mathcal{L}^r(X, \mu)$ with $\| \prod_{k=1}^m f_k\|_{\mathcal{L}^r} \le \prod_{k=1}^m \|f_k\|_{\mathcal{L}^{p_k}}$.

Exercise 40 (4 points). Let $\mu\colon \mathscr{P}(\mathbb{R}) \to [0, +\infty]$ be a Borel outer measure such that $\mu(\mathbb{R}) = 1$. Prove that

$$\mathscr{L}^1(E) = \int_{\mathbb{R}} \mu(x + E)\, dx$$

for all Borel sets $E \subset \mathbb{R}$.

Exercise 41 (4 points). Let $(X, \mathscr{A}, \mu)$ be a measurable space with $\mu(X) < +\infty$. Let $f\colon X \to [0, +\infty]$ be a μ-measurable function, not μ-a.e. equal to the null function, such that $f \in \mathcal{L}^p(X, \mu)$ for all $p \in [1, +\infty)$. Prove that

$$\lim_{p \to +\infty} \frac{\|f\|_{\mathcal{L}^{p+1}}^{p+1}}{\|f\|_{\mathcal{L}^p}^p} = \|f\|_{\mathcal{L}^\infty}.$$

Exercise 42 (4 points). Let $\varphi\colon [0, 1] \to \mathbb{R}$ be continuous function such that $\varphi(x) > 0$ for all $x \in [0, 1]$. Prove that

$$\lim_{p \to +\infty} \left(\int_{[0,1]} |f|^p \, \varphi \, d\mathscr{L}^1 \right)^{\frac{1}{p}} = \|f\|_{\mathcal{L}^\infty}$$

for all $f \in \mathcal{L}^1([0, 1], \mathscr{L}^1)$.

Bonus (2 points). Find $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f \in \bigcap_{p \in [1, +\infty)} \mathcal{L}^p(\mathbb{R}^n, \mathscr{L}^n) \setminus \mathcal{L}^\infty(\mathbb{R}^n, \mathscr{L}^n)$.

Exercises for week 12

Giorgio Stefani

Exercise 43 (4 points)**.** Let $(X, \mathscr{A}, \mu)$ be a measurable space. Let $(A_k)_{k\in\mathbb{N}} \subset \mathscr{A}$ be a sequence of pairwise disjoint sets such that $\mu(A_k) \in (0, +\infty)$ for all $k \in \mathbb{N}$. Let $1 \le p < q < +\infty$.

(i) [2 points] Assume that there exists a subsequence $(A_{k_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to+\infty} \mu(A_{k_j}) = 0$. Prove that $L^p(X,\mu) \setminus L^q(X,\mu) \neq \varnothing$.

(ii) [2 points] Assume that there exists a subsequence $(A_{k_j})_{j\in\mathbb{N}}$ such that $\inf_{j\in\mathbb{N}} \mu(A_{k_j}) > 0$. Prove that $L^q(X,\mu) \setminus L^p(X,\mu) \neq \varnothing$.

Exercise 44 (4 points)**.** Let $(X, \mathscr{A}, \mu)$ be a measurable space. Prove that the following statements are equivalent.

(i) There exists a sequence $(A_k)_{k\in\mathbb{N}} \subset \mathscr{A}$ such that $\mu(A_k) > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to+\infty} \mu(A_k) = 0$.

(ii) There exists a sequence $(B_k)_{k\in\mathbb{N}} \subset \mathscr{A}$ such that $0 < \mu(B_k) \le \frac{1}{2^k}$ for all $k \in \mathbb{N}$.

(iii) $L^1(X,\mu) \setminus L^\infty(X,\mu) \neq \varnothing$.

(iv) There exists a sequence $(C_k)_{k\in\mathbb{N}} \subset \mathscr{A}$ of pairwise disjoint sets such that $0 < \mu(C_k) \le \frac{1}{2^k}$ for all $k \in \mathbb{N}$.

Exercise 45 (4 points)**.** Let $p \in [1, +\infty]$. Prove that $(L^p(\mathbb{R}^n, \mathscr{L}^n), \|\cdot\|_p)$ is a Hilbert space if [2 points] and only if [2 points] $p = 2$.

Exercise 46 (4 points)**.** Let $(X, \mathscr{A}, \mu)$ be a measurable space and let $p \in [1, +\infty)$. Assume that $(f_k)_{k\in\mathbb{N}} \subset L^p(X,\mu)$ and $f \in L^p(X,\mu)$ are such that

(i) $\lim_{k\to+\infty} f_k(x) = f(x)$ for μ-a.e. $x \in X$;

(ii) $\lim_{k\to+\infty} \|f_k\|_p = \|f\|_p$.

Prove that $\lim_{k\to+\infty} \|f_k - f\|_p = 0$.

Bonus (2 points)**.** Let $(X, \mathscr{A}, \mu)$ be a measurable space. Let $1 \le p < q < r \le +\infty$. Prove that $L^q(X,\mu) \subset L^p(X,\mu) + L^r(X,\mu)$, that is, if $f \in L^q(X,\mu)$, then $f = f_1 + f_2$ for some $f_1 \in L^p(X,\mu)$ and $f_2 \in L^r(X,\mu)$.

Exercises for week 13

Giorgio Stefani

Exercise 47. Let $(X, \mathscr{A}, \mu)$ be a measurable space and let $p \in [1, +\infty)$. Let $(f_k)_{k\in\mathbb{N}} \subset L^p(X, \mu)$ be such that $f_k \to f$ in $L^p(X, \mu)$ as $k \to +\infty$ for some $f \in L^p(X, \mu)$. Prove that $f_k \to f$ in measure as $k \to +\infty$.

Exercise 48. Let $(X, \mathscr{A}, \mu)$ be a measurable space such that $\mu(X) = 1$. Let $\varphi\colon [0, +\infty) \to [0, +\infty)$ be a continuous, convex and monotone increasing function and define $\varphi(+\infty) = +\infty$. Prove that

$$\varphi\left(\int_X f \, d\mu\right) \le \int_X \varphi(f) \, d\mu$$

for all non-negative measurable functions $f\colon X \to [0, +\infty]$.

Exercise 49. Let $(X, \mathscr{A}, \mu)$ be a measurable space such that $\mu(X) < +\infty$. Let $p \in (1, +\infty)$ and let $(f_k)_{k\in\mathbb{N}} \subset L^p(X, \mu)$ be such that $M = \sup_{k\in\mathbb{N}} \|f_k\|_p < +\infty$. Assume that $f_k \to f$ μ-a.e. in X as $k \to +\infty$ for some measurable function $f\colon X \to [-\infty, +\infty]$.

(i) Prove that $f \in L^p(X, \mu)$ with $\|f\|_p \le M$.

(ii) Prove that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$E \in \mathscr{A} \text{ with } \mu(E) < \delta \implies \sup_{k\in\mathbb{N}} \int_E |f_k - f|^q \, d\mu < \varepsilon.$$

(iii) Prove that $f_k \to f$ in $L^q(X, \mu)$ as $k \to +\infty$ for all $q \in [1, p)$.

(iv) Prove that one cannot take $q = p$ in the previous point with a counterexample.

Exercise 50. Let $(X, \mathscr{A}, \mu)$ be a measurable space such that $\mu(X) < +\infty$. Let $p \in (1, +\infty)$ and let $(f_k)_{k\in\mathbb{N}} \subset L^p(X, \mu)$ be such that $M = \sup_{k\in\mathbb{N}} \|f_k\|_p < +\infty$. Assume that $f_k \to f$ in measure as $k \to +\infty$ for some measurable function $f\colon X \to [-\infty, +\infty]$.

(i) Prove that $f \in L^p(X, \mu)$ with $\|f\|_p \le M$.

(ii) Prove that $\lim_{k\to+\infty} \int_{\{|f_k - f| \ge \varepsilon\}} g \, d\mu = 0$ for all $\varepsilon > 0$ and all $g \in L^1(X, \mu)$.

(iii) Prove that $f_k \to f$ weakly in $L^p(X, \mu)$ as $k \to +\infty$.

(iv) Prove that one cannot take $p = 1$ in the previous point with a counterexample.

Extra exercises

Giorgio Stefani

Exercise 51. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Assume that

$$\sup_{k\in\mathbb{N}} \int_{\mathbb{R}} |x|^k \, |f(x)| \, dx < +\infty.$$

Is it true that $f(x) = 0$ for $\mathscr{L}^1$-a.e. $x \in \mathbb{R} \setminus [-1,1]$? If yes, then prove it. If no, then give a counterexample.

Exercise 52. Let $(X, \mathscr{A}, \mu)$ be a measurable space. Let $f, f_k \in L^1(X,\mu)$, $k \in \mathbb{N}$, be such that $f_k \to f$ strongly in $L^1(X,\mu)$ as $k \to +\infty$. Let $(A_k)_{k\in\mathbb{N}} \subset \mathscr{A}$ be such that $\mu(A_k) \to 0$ as $k \to +\infty$. Prove that $\lim\limits_{k\to+\infty} \int_{A_k} f_k \, d\mu = 0$.

Exercise 53. Let $(X, \mathscr{A}, \mu)$ be a measurable space with $\mu(X) < +\infty$. Let $f, f_k\colon X \to [-\infty,+\infty]$, $k \in \mathbb{N}$, be μ-measurable functions. Prove that the following two statements are equivalent.

(A) $f_k \to f$ in measure as $k \to +\infty$.
(B) $\lim\limits_{k\to+\infty} \int_X 1 \wedge |f_k - f| \, d\mu = 0$.

Exercise 54. Let $(f_k)_{k\in\mathbb{N}} \subset L^2([0,1])$ be such that $\|f_k\|_{L^2([0,1])} \le \sqrt{2}$ for all $k \in \mathbb{N}$. Assume that $f_k \to 0$ in measure as $k \to +\infty$.

(i) Fix $\eta > 0$ and define $g_{\eta,k} = \mathbf{1}_{\{|f_k|>\eta\}}$ for all $k \in \mathbb{N}$. Prove that

$$\lim_{k\to+\infty} \int_{[0,1]} g_{\eta,k} \, \varphi \, dx = 0$$

for all $\varphi \in L^1([0,1])$.
(ii) Prove that $f_k \to 0$ weakly in $L^2([0,1])$ as $k \to +\infty$.

Exercise 55. Let $(X, \mathscr{A}, \mu)$ be a σ-finite measurable space. Let $f\colon X \to [0,+\infty]$ be a non-negative μ-measurable function. Prove that, if $\mu(\{x \in X : f(x) > t\}) \ge \frac{1}{1+t}$ for all $t \ge 0$, then $f \notin L^1(X,\mu)$.

Exercise 56. Let $(f_k)_{k\in\mathbb{N}} \subset L^3([0,1])$ be such that $\|f_k\|_{L^3([0,1])} \le 2$ for all $k \in \mathbb{N}$. Assume that $f_k \to 0$ $\mathscr{L}^1$-a.e. on $[0,1]$ as $k \to +\infty$.

(i) Prove that $f_k \in L^2([0,1])$ for all $k \in \mathbb{N}$.
(ii) Let $g_k = f_k^2$ for all $k \in \mathbb{N}$. Prove that $(g_k)_{k\in\mathbb{N}} \subset L^1([0,1])$ is uniformly integrable.
(iii) Prove that $f_k \to 0$ strongly in $L^2([0,1])$ as $k \to +\infty$.

Exercise 57. Let $E, E_k \subset [0,1]$, $k \in \mathbb{N}$, be Lebesgue measurable sets. Prove that the following two statements are equivalent for any fixed $p \in [1,+\infty)$.

(A) $\mathbf{1}_{E_k} \to \mathbf{1}_E$ strongly in $L^p([0,1])$ as $k \to +\infty$.
(B) $\mathbf{1}_{E_k} \to \mathbf{1}_E$ weakly in $L^p([0,1])$ as $k \to +\infty$.