CHRONOLOGICAL ALGEBRAS AND NONSTATIONARY VECTOR FIELDS

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A calculus is developed, reflecting the most general group-theoretic properties of flows, defined by nonstationary differential equations on manifolds.

INTRODUCTION

The basic object of study in the present paper is flows defined by nonstationary ordinary differential equations on a smooth manifold. In it there is developed a calculus reflecting the most general group-theoretic properties of such flows. The development of this calculus was basically stimulated by variational problems of control theory, especially the many attempts to extend the maximum principle of L. S. Pontryagin to singular problems of control (cf. [1, 2, 4, 7-10]). However, we hope that it turns out to be useful also for solving certain geometric problems.

As is known, the main difficulty in expressing a flow in terms of the nonstationary vector field defining it is the circumstance that the fields at different moments of time do not commute. In describing the phenomena arising here an important role is played by certain nonstandard algebraic constructions. We call the corresponding algebras "chronological," using a term which is used in physics in the analogous nonstationary situations.

Certain aspects of the "chronological" calculus are discussed in [3]. Our account is independent of this paper, but the proofs of certain assertions which are in [3] are omitted, substituting their corresponding references.

The first section of this paper is devoted to the description of the necessary algebraic constructions, in the second section we establish the connection between the algebraic objects introduced earlier and the group of flows on a manifold.

1. Chronological Algebras

1. Definition and Examples of ch-Algebras. Let $\mathfrak{A}$ be some algebra over a fixed field k, i.e., a vector space over K with a bilinear product

$$(x, y)\rightarrow xy, \quad x, \ y\in\mathfrak{A}.$$

To each $x\in\mathfrak{A}$ we associate a linear transformation L_x of the vector space $\mathfrak{A}$, acting by the rule $L_x y = xy$. We denote by $[x, y]$ the commutator of the elements x and y in our algebra, $[x, y] = xy - yx$.

On the other hand, $[L_x, L_y] = L_x L_y - L_y L_x$ is the usual commutator of linear transformations of the vector space $\mathfrak{A}$, turning the collection of all linear transformations $\mathscr{L}(\mathfrak{A})$ into a Lie algebra over K.

In what follows, we shall always mean by $[\mathfrak{A}]$ the algebra which one gets from the given algebra $\mathfrak{A}$ if one replaces the multiplication xy by the commutation operation $[x, y] = xy - yx$.

Definition. The algebra $\mathfrak{A}$ is called a chronological algebra, or for short, a ch-algebra, if $\forall x, y\in\mathfrak{A}$ one has

$$L_{[x,y]} = [L_x, L_y], \tag{1.1}$$

i.e., the correspondence $x \to L_x$ is a homomorphism of the algebra $[\mathfrak{A}]$ into the Lie algebra $\mathscr{L}(\mathfrak{A})$.

Equation (1.1) is equivalent with the following identity, which singles out chronological algebras from the set of all algebras more directly:

$$x(yz) - y(xz) = (xy - yx)z \quad \forall x, \ y, \ z\in\mathfrak{A}.$$

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In particular, any associative algebra is a chronological algebra.

The condition of associativity of the multiplication in an arbitrary algebra can be written in the form $L_{xy} = L_x L_y$. Hence, in order that the chronological algebra $\mathfrak{A}$ be associative, it is necessary and sufficient that one have

$$L_{xx} = L_x L_x \quad \forall x \in \mathfrak{A}.$$

In general, a ch-algebra does not necessarily contain a unit element; however, to any such algebra one can adjoin a unit. In more detail, let $\mathfrak{A}$ be a ch-algebra; we consider the direct sum of the vector space $\mathfrak{A}$ with the field of scalars K and we define multiplication in the space $\mathfrak{A} \oplus K$ by the formula

$$(x+\alpha)(y+\beta) = xy + \beta x + \alpha y + \alpha\beta \quad \forall x,\ y \in \mathfrak{A},\ \alpha,\ \beta \in K.$$

One can verify directly that $\mathfrak{A} \oplus K$ with this multiplication is a ch-algebra, and the unit of the field K is the unit element of this algebra, $1(x + \alpha) = (x + \alpha)1 = x + \alpha$. A more important property of chronological algebras is the following.

Proposition 1.1. If $\mathfrak{A}$ is a ch-algebra, then $[\mathfrak{A}]$ is a Lie algebra. In other words, the commutation operation defines in an arbitrary chronological algebra the structure of a Lie algebra.

The proof is simple. The definition of ch-algebra says that the correspondence $x \to L_x$ is a homomorphism of $[\mathfrak{A}]$ into the Lie algebra $\mathscr{L}(\mathfrak{A})$. We adjoin to $\mathfrak{A}$ a unit with the help of the construction described above, so the correspondence $x \to L_x$ becomes an injective homomorphism of the algebra $[\mathfrak{A}]$ into the Lie algebra $\mathscr{L}(\mathfrak{A} \oplus K)$.

Consequently, $[\mathfrak{A}]$ is isomorphic with some subalgebra of the Lie algebra $\mathscr{L}(\mathfrak{A} \oplus K)$, which means it itself is a Lie algebra.

Examples of ch-algebras:

a) we have already remarked that an arbitrary associative algebra is a ch-algebra. Now we shall show that if the given ch-algebra $\mathfrak{A}$ is commutative, i.e., satisfies the condition

$$xy = yx \quad \forall x,\ y \in \mathfrak{A},$$

then it is necessarily associative.

In fact, since $[x, y] = 0$, we get from the definition of ch-algebra that $L_x L_y = L_y L_x$ $\forall x,\ y$. Consequently, $\forall x,\ y,\ z \in \mathfrak{A}$

$$x(yz) = x(zy) = L_x L_z y = L_z L_x y = z(xy) = (xy)z.$$

b) Let us assume that the characteristic of the field K is not equal to 2. Let $\mathfrak{A}$ be a skew-symmetric ch-algebra, i.e., a ch-algebra in which one has the identity

$$xy = -yx \quad \forall x,\ y \in \mathfrak{A}.$$

Then $[x, y] = xy - yx = 2xy$. Consequently, $\mathfrak{A}$ is isomorphic with the Lie algebra $[\mathfrak{A}]$; in particular, the ch-algebra $\mathfrak{A}$ itself is simultaneously a Lie algebra. From the Jacobi identity in the algebra $\mathfrak{A}$ follows the equality $L_{(xy)} = [L_x, L_y]$. On the other hand, as we know, $[L_x, L_y] = L_{[x,y]} = 2L_{(xy)}$. Thus, $L_{xy} = 2L_{xy}$. Consequently, $L_{xy} = 0$ for $\forall x,\ y \in \mathfrak{A}$. Finally, we get: a skew-symmetric algebra $\mathfrak{A}$ is a ch-algebra if and only if the product of any three elements is equal to zero,

$$(xy)z = 0 \quad \forall x,\ y,\ z \in \mathfrak{A}.$$

c) The following example is the most important for us. In contrast with the preceding ones it is not formulated purely algebraically. Here we assume that K is either the field of complex numbers or the field of real numbers.

Let $\mathfrak{L}$ be some finite-dimensional Lie algebra. We denote by $\mathfrak{L}_T$ the vector space of all absolutely continuous mappings $x: [0, \infty) \to \mathfrak{L}$, vanishing for the zero value of the argument. Thus, $\mathfrak{L}$ consists of those elements $x(t)$ of the algebra $\mathfrak{L}$ which depend absolutely continuously on the time t and are such that $x(0) = 0$.

The space $\mathfrak{L}_T$ is, of course, a Lie algebra,

$$[x, y](t) = [x(t), y(t)] \quad \forall x,\ y \in \mathfrak{L}_T.$$

We note that the value of the Lie bracket $[x, y]$ at time t depends only on the values of x and y at this moment of time t, and the "prehistory" here is not considered at all. Now we define in $\mathfrak{L}_T$ another binary operation, the result of applying which to the pair x, y, calculated at time t, depends essentially on the values $x(\tau)$, $y(\tau)$ on the entire segment $0 \le \tau \le t$. Namely, we set

$$(x * y)(t) = \int_0^t \left[x(\tau), \frac{d}{d\tau} y(\tau) \right] d\tau.$$

Integrating by parts, taking account of the skew-symmetry of the Lie bracket, leads to the formula

$$x * y - y * x = [x, y].$$

Thus, the Lie bracket can be reconstructed from the operation "*." Actually, this operation defines a ch-algebra structure on the space $\mathfrak{L}_T$. In fact,

$$([x, y] * z)(t) = \int_0^t \left[[x(\tau), y(\tau)], \frac{d}{d\tau} z(\tau) \right] d\tau$$

$$= \int_0^t \left[\left[x(\tau), \frac{d}{d\tau} z(\tau) \right], y(\tau) \right] d\tau + \int_0^t \left[x(\tau), \left[y(\tau), \frac{d}{d\tau} z(\tau) \right] \right] d\tau = x * (y * z)(t) - y * (x * z)(t).$$

Remark. Actually, we need a more general variant of example c) when the algebra $\mathfrak{L}$ is infinite-dimensional. However, in the infinite-dimensional case one requires a special refinement of the concept of continuity, so in this section we restrict the description to the finite-dimensional case only.

Later we shall need to consider graded algebras, so we recall the corresponding definition. By a grading on a given algebra $\mathfrak{A}$, we mean a representation of $\mathfrak{A}$ as a direct sum of subspaces, $\mathfrak{A} = \sum_{n=1}^{\infty} \mathfrak{A}_n$, satisfying the condition $\mathfrak{A}_i \mathfrak{A}_j \subset \mathfrak{A}_{i+j}$ for i, j = 1, 2, ...† An algebra with a given grading will be called a graded algebra. Each subspace $\mathfrak{A}_n$ will be called the homogeneous component of degree n of the graded algebra $\mathfrak{A}$, or, for short, the n-th component. We say that the element x of $\mathfrak{A}$ has degree n if $x \in \left(\sum_{i=1}^{n} \mathfrak{A}_i \right) \setminus \left(\sum_{i=1}^{n-1} \mathfrak{A}_i \right)$ (the subtraction is set-theoretic). If $\mathfrak{A} = \sum_{n=1}^{\infty} \mathfrak{A}_n$ is a graded algebra, then the algebra $[\mathfrak{A}]$ is also graded, where the homogeneous components are the same subspaces $\mathfrak{A}_n$.

2. Free ch-Algebras. We begin with some standard definitions. The ch-algebras considered below, in general, do not have units. All the algebras are taken over one and the same fixed field K.

Let S be some set. The ch-algebra $\mathfrak{A}$ is called the free ch-algebra with generating set S if there exists an imbedding $i: S \to \mathfrak{A}$ such that for any ch-algebra $\mathfrak{B}$ and any mapping $f: S \to \mathfrak{B}$ one can find a unique homomorphism $f_*: \mathfrak{A} \to \mathfrak{B}$ for which one has a commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{f_*} & \\ i \uparrow & \searrow f & \\ S & \to & \mathfrak{B}. \end{array}$$

Free ch-algebras with generating set S exist, e.g., as the quotient-algebra of the free (nonassociative) algebra with generating set S by the two-sided ideal generated by elements of the form

$$x(yz) - y(xz) - (xy - yx)z. \tag{1.2}$$

Here x, y, z are arbitrary elements of the free nonassociative algebra.

From "abstract nonsense" the uniqueness up to isomorphism of the free ch-algebra with given generating set S follows instantly.

Further, it is obvious that any ch-algebra $\mathfrak{B} \supset S$, which is generated by elements of the set S, is isomorphic to a quotient-algebra of the corresponding free algebra by some ideal.

We denote by A_S the free (nonassociative) algebra, and by $\mathfrak{A}_S$ the free ch-algebra with generating set S. The algebra A_S has a natural grading, under which the elements of the first degree are precisely the linear combinations of the elements of the set S. The algebra $\mathfrak{A}_S$ can be obtained by factorizing A_S by (1.2). Since this relation is homogeneous, the factorization induces the structure of a graded algebra on $\mathfrak{A}_S$, where the

†More correctly this should be called a positive grading, but no other kind of grading will occur for us.

elements of the first degree are again the linear combinations of elements of the set S (we identify S with the image of S under the factorization). In addition, the algebra $\mathfrak{A}_S$ is generated by its elements of the first degree, just as A_S is.

We recall that $[\mathfrak{A}_S]$ is the Lie algebra with grading, whose elements are the elements of $\mathfrak{A}_S$, and the commutator is defined by

$$[x, y]=xy-yx \quad \forall x, y\in\mathfrak{A}_S.$$

Let $U[\mathfrak{A}_S]$ be the universal enveloping algebra of the Lie algebra $[\mathfrak{A}_S]$. The grading in $[\mathfrak{A}_S]$ uniquely defines a grading of the algebra $U[\mathfrak{A}_S]$. Consequently, $U[\mathfrak{A}_S]$ is a graded associative algebra.

The correspondence $x\mapsto L_x$ $(x\in\mathfrak{A})$ is a representation of the Lie algebra $[\mathfrak{A}_S]$ by linear transformations of the algebra $\mathfrak{A}_S$, considered simply as a vector space. This representation extends uniquely to a representation of the associative algebra $U[\mathfrak{A}_S]$. Such a representation we shall also denote by the letter L; namely, an element $u\in U[\mathfrak{A}_S]$ corresponds to a linear operator L_u.

Below, we denote multiplication in the associative algebra $U[\mathfrak{A}_S]$ by a small circle "$\circ$," in contrast with multiplication in the ch-algebra $\mathfrak{A}_S$, for which we use no special notation.

LEMMA 1.1. The linear span of the set

$$L_{U[\mathfrak{A}_S]}S=\{L_u s \mid u\in U[\mathfrak{A}_S], s\in S\}\subset\mathfrak{A}_S$$

coincides with the entire algebra $\mathfrak{A}_S$.

Proof. If the element u of the graded algebra $U[\mathfrak{A}_S]$ has degree $k \geq 0$, then for any $s \in S$ the element $L_u s$ of the ch-algebra $\mathfrak{A}_S$ has degree $k + 1$. Hence it is natural to use induction on k. Namely, we shall show that for any $k \geq 0$ the elements of degree $k + 1$ of the graded algebra $\mathfrak{A}_S$ lie in the linear span of the set $L_{U[\mathfrak{A}_S]}S$.

For $k = 0$ the assertion is obvious. Let us assume that it is true for all k less than a given $n > 0$, and we shall establish it in the case $k = n$.

It is clear that all elements of the algebra $\mathfrak{A}_S$ of degree $n + 1$ lie in the linear span of elements of the form xy, where $xy\in\mathfrak{A}_S$, while the degree of both x and y does not exceed n. Using the inductive hypothesis, we get that $y=\sum_{i=1}^{l} L_{u_i}s_i$, where $u_i\in U[\mathfrak{A}_S]$. Thus,

$$xy=L_xy=L_x\sum_{i=1}^{l}L_{u_i}s_i=\sum_{i=1}^{l}L_xL_{u_i}s_i=\sum_{i=1}^{l}L_{(x\circ u_i)}s_i.$$

In this calculation we consider x as an element of the Lie algebra $[\mathfrak{A}_S]\subset U[\mathfrak{A}_S]$ and we use the fact that L is a representation of the associative algebra $U[\mathfrak{A}_S]$. The lemma is proved.

We quickly get a much more precise result.

We denote by $\bar{S}$ the linear span of the set S in $\mathfrak{A}_S$. It is clear that the set S forms a basis for the vector space $\bar{S}$. We consider the tensor product $U[\mathfrak{A}_S]\otimes\bar{S}$ of the vector spaces $U[\mathfrak{A}_S]$ and S. We define the structure of an algebra on $U[\mathfrak{A}_S]\otimes\bar{S}$ by defining multiplication with the help of the formula

$$(u_1\otimes s_1)(u_2\otimes s_2)=((L_{u_1}s_1)\circ u_2)\otimes s_2, \quad \forall u_1, u_2\in U[\mathfrak{A}_S], \; s_1, s_2\in\bar{S}.$$

We denote the algebra so obtained by $\mathfrak{B}_S$. It is turned into a graded algebra if we set the degree of $u\otimes s$ equal to the degree of u plus 1.

We assert that $\mathfrak{B}_S$ is a ch-algebra.

Here is the verification:

$$(u_1\otimes s_1)((u_2\otimes s_2)(u_3\otimes s_3))-(u_2\otimes s_2)((u_1\otimes s_1)(u_3\otimes s_3))=((L_{u_1}s_1)\circ(L_{u_2}s_2)-(L_{u_2}s_2)\circ(L_{u_1}s_1))\circ u_3\otimes s_3=[L_{u_1}s_1, L_{u_2}s_2]\circ u_3\otimes s_3.$$

On the other hand,

$$((u_1\otimes s_1)(u_2\otimes s_2)-(u_2\otimes s_2)(u_1\otimes s_1))(u_3\otimes s_3)=(L_{(L_{u_1}s_1)\circ u_2}s_2-L_{(L_{u_2}s_2)\circ u_1}s_1)\circ u_3\otimes s_3$$
$$=(L_{(L_{u_1}s_1)}\circ L_{u_2}s_2-L_{(L_{u_2}s_2)}\circ L_{u_1}s_1)\circ u_3\otimes s_3=((L_{u_1}s_1)(L_{u_2}s_2)-(L_{u_2}s_2)(L_{u_1}s_1))\circ u_3\otimes s_3.$$

THEOREM 1.1. The graded ch-algebra $\mathfrak{B}_S$ is isomorphic with the universal ch-algebra.

Proof. We define a map $f: S \to \mathfrak{B}_S$ by the rule $f(s) = 1 \otimes s$.

From the definition of a free ch-algebra it follows that there exists a unique homomorphism of ch-algebras $f_*: \mathfrak{A}_S \to \mathfrak{B}_S$, satisfying the condition

$$f_*(s) = 1 \otimes s \quad \forall s \in S \subset \mathfrak{A}_S.$$

We shall show that f_* is an isomorphism. For this, we establish the identity

$$f_*(L_u s) = u \otimes s, \quad \forall u \in U[\mathfrak{A}_S], \quad s \in S. \tag{1.3}$$

From (1.3) it follows immediately that the map f_* is surjective and, considering Lemma 1, that it is also injective.

Equation (1.3) will be proved by induction on the degree of the element u. For degree zero it is valid. Let us assume that this identity holds for all elements of the algebra $U[\mathfrak{A}_S]$ having degree less than a given $n > 0$, and let the degree of u be equal to n.

The element u can be represented in the form

$$u = \sum_{i=1}^{l} x_i \circ v_i, \text{where } x_i \in [\mathfrak{A}_S],$$

and the degree of x_i and v_i is less than n, $i = 1, \ldots, l$.

Further, we get from Lemma 1.1 that

$$x_i = \sum_{j=1}^{l_i} L_{w_{ij}} t_{ij}, \quad w_{ij} \in U[\mathfrak{A}_S], \quad t_{ij} \in S.$$

Finally, $u = \sum_{i,j} (L_{w_{ij}} t_{ij}) \circ v_i$.

By virtue of the linearity of (1.3), it suffices to verify it for each summand separately, i.e., one can assume that $u = (L_w t) \circ v$, while the degree of v and w is less than n.

We have

$$f_*(L_u s) = f_*(L_{L_w t} L_v s) = f_*((L_w t)(L_v s)) = f_*(L_w t) f_*(L_v s).$$

We have used the fact that f_* is a homomorphism of ch-algebras. Now, using the inductive hypothesis for each of the two factors separately,

$$f_*(L_w t) f_*(L_v s) = (w \otimes t)(v \otimes s) = (L_w t) \circ v \otimes s = u \otimes s.$$

Equation (1.3) and with it the theorem are proved.

Theorem 1.1 makes it possible to construct an additive basis of the algebra $\mathfrak{A}_S$. By an additive basis of an arbitrary algebra we mean a basis of the vector space which one gets from this algebra if one "forgets" the operation of multiplication.

Let the set $\mathcal{U}_n \subset U[\mathfrak{A}_S]$ form a basis for the n-th component of the graded algebra $U[\mathfrak{A}_S]$, $n = 1, 2, \ldots$.

From the proof of Theorem 1 [cf., in particular, (1.3)] it follows that the set $\{L_u s \mid u \in \mathcal{U}_n, \ s \in S\}$ is a basis of the $(n + 1)$-st component of the graded ch-algebra $\mathfrak{A}_S$.

Further, the Poincaré–Birkhoff–Witt theorem gives a method for constructing an additive basis for the universal enveloping algebra $U[\mathfrak{A}_S]$, and here in constructing a basis for the n-th component one uses elements of degree not higher than n of the algebra $[\mathfrak{A}_S]$. Finally, we get the following corollary.

COROLLARY 1.1. Let the set $\mathcal{A}_n \subset \mathfrak{A}_S$ form a basis for the linear subspace of the algebra $\mathfrak{A}_S$ consisting of elements of degree not higher than n. Let us assume in addition that $\mathcal{A}_n$ is a homogeneous basis (i.e., each element of $\mathcal{A}_n$ lies in some homogeneous component of the graded algebra $\mathfrak{A}_S$) and that the set $\mathcal{A}_n$ is ordered in some way.

Let the symbol $\preccurlyeq$ denote the order relation (not greater than) in $\mathcal{A}_n$, and deg y be the degree of the element y of $\mathcal{A}_n$. Then the set

$$\{x = L_{y_1}\circ\ldots\circ L_{y_{k_x}} s \mid y_i \in \mathcal{A}_n,\ i=1,\ldots,k_x,\ s\in S,\ y_1 \preccurlyeq y_2 \preccurlyeq \ldots \preccurlyeq y_{k_x} \text{ and} \sum_{i=1}^{k_x} \deg y_i = n\}$$

forms a basis of the (n + 1)-st component of the algebra $\mathfrak{A}_S$.

Thus, a method is indicated for the explicit construction of a basis of the (n + 1)-st component of the algebra $\mathfrak{A}_S$ if there is already known a basis for the components of lower degree. A basis for the first component is obviously S.

In correspondence with the procedure described, we find a basis for the first four components in the case when the set S consists of one element, S = {s}:

degree 1) $y_1 = s$;

degree 2) $y_2 = L_{y_1} s = s^2$;

degree 3) $y_3 = L_{y_2} s = s^2 s$, $y_4 = L_{y_1} L_{y_1} s = ss^2$;

degree 4) $y_5 = L_{y_3} s = (s^2 s)s$, $y_6 = L_{y_4} s = (ss^2)s$, $y_7 = L_{y_1} L_{y_2} s = s(s^2 s)$, $y_8 = L_{y_1} L_{y_1} L_{y_1} s = s(ss^2)$.

We shall use this table again.

Let m be a positive integer. We denote by $\mathfrak{A}_m$ the free ch-algebra with m generators, i.e., the algebra $\mathfrak{A}_S$ in the case when S consists of m elements, $\mathfrak{A}_m \overset{\text{def}}{=} \mathfrak{A}_{\{s_1,\ldots,s_m\}}$. The homogeneous components of the graded algebra $\mathfrak{A}_m$ are finite-dimensional vector spaces. Now we get a recursion formula for calculating their dimensions.

We denote by $b_m^{(n)}$ the dimension of the n-th component of the algebra $\mathfrak{A}_m$.

From Corollary 1.1, by standard arguments usually used in combinatorial problems connected with partitioning numbers into summands, we get that $b_m^{(n+1)}$ is equal to the coefficient of t^n in the Taylor series expansion of the function of t: $m \sum_{k=1}^{n} (1-t^k)^{-b_m^{(k)}}$. Whence follows the identity in formal series in the variable t

$$\sum_{n=0}^{\infty} b_m^{(n+1)} t^n = m \prod_{n=1}^{\infty} (1-t^n)^{-b_m^{(n)}}. \tag{1.4}$$

Equation (1.4) allows one to calculate successively the numbers $b_m^{(n)}$. We write the first few:

$$b_m^{(1)} = m,\ b_m^{(2)} = m^2,\ b_m^{(3)} = \frac{m^2}{2}(3m+1),\ b_m^{(4)} = \frac{m^2}{3}(8m^2+3m+1).$$

We introduce the notation $\varphi_m(t) = \sum_{n=0}^{\infty} b_m^{(n+1)} t^n$, which is the generating series for the numbers $b_m^{(n)}$. Using his series, one can transform (1.4) somewhat. Taking logarithms gives

$$\ln\left(\frac{\varphi_m(t)}{m}\right) = -\sum_{n=1}^{\infty} b_m^{(n)} \ln(1-t^n) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{b_m^{(n)}}{k} t^{nk} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} b_m^{(n)} t^{nk} = \sum_{k=1}^{\infty} \frac{t^k}{k} \varphi_m(t^k).$$

Thus,

$$\varphi_m(t) = m e^{\sum_{k=1}^{\infty} \frac{t^k}{k} \varphi_m(t^k)}$$

Again let S be an arbitrary set. We proceed to a more detailed study of the Lie algebra $[\mathfrak{A}_S]$. We note, rstly, that the algebra $U[\mathfrak{A}_S]$, like any universal enveloping algebra, has a natural filtration. Namely, we enote by $U_n[\mathfrak{A}_S]$ the linear subspace of $[\mathfrak{A}_S]$ generated by elements of the form $y_1 \circ \ldots \circ y_k$, $k \le n$, $y_i \in [\mathfrak{A}_S]$ = 1, . . . , k). In particular, $U_0[\mathfrak{A}_S] = K$, $U_1[\mathfrak{A}_S]$ consists of the elements of the form $\alpha + y$, where $\alpha \in K$, $y \in [\mathfrak{A}_S]$. he increasing sequence of subspaces $U_n[\mathfrak{A}_S]$ also forms a filtration of the algebra $U[\mathfrak{A}_S]$.

This filtration of $U[\mathfrak{A}_S]$ generates a filtration of the algebra $\mathfrak{A}_S$ in the following way. Let $[\mathfrak{A}_S]_n$ be a near span of elements of the form $L_u s$, where $u \in U_n[\mathfrak{A}_S]$, $s \in S$. The algebra $\mathfrak{A}_S$ coincides, by virtue of Lemma 1, with the union of the increasing sequence of subspaces $[\mathfrak{A}_S]_n$, which thus give a filtration of it.

In particular, $[\mathfrak{A}_S]_0 = \bar{S}$, the subspace $[\mathfrak{A}_S]_1$ is the linear span of the elements of the set S and the elements the form xs, where $x \in \mathfrak{A}_S$, $s \in S$.

Proposition 1.2. The subspace $[\mathfrak{A}_S]_1 \subset [\mathfrak{A}_S]$ generates the Lie algebra $[\mathfrak{A}_S]$. In other words, any element of $\mathfrak{A}_S$ can be expressed in terms of elements of $[\mathfrak{A}_S]_1$ using only the operations of commutation and addition.

Proof. We note, first of all, that the assertion of Proposition 1.2 is equivalent with the fact that the subspace $[\mathfrak{A}_S]_1 \oplus K \subset U[\mathfrak{A}_S]$ generates the associative algebra $U[\mathfrak{A}_S]$, i.e., any element of $U[\mathfrak{A}_S]$ can be expressed in terms of elements of $[\mathfrak{A}_S]_1$ and scalars with the help of the associative multiplication "$\circ$" and addition.

To prove Proposition 1.2, we use double induction. The first induction is on the degree of the elements.

Thus, let us assume that for elements of the algebra $[\mathfrak{A}_S]$ of degree no higher than n, everything is proved. It is required to show that any element of the form $L_u s$, where $u \in U[\mathfrak{A}_S]$, $s \in S$, where u has degree n, can be represented in the form of a sum of commutators of elements of degree $\leq$ n plus some element of $[\mathfrak{A}_S]_1$.

Since for degrees $\leq$ n the assertion of Proposition 1.2 is assumed to be valid, the element u can be represented in the form of a sum of elements of the form $x_1 \circ \ldots \circ x_k$, where $x_i \in [\mathfrak{A}_S]_1$. Hence, without loss of generality, one can assume that $u = x_1 \circ v$, $x_1 \in [\mathfrak{A}_S]_1$.

The second induction is on the degree of x_1.

a) Let us assume that x_1 has the first degree, so one can assume that $x_1 = s_1 \in S$. In this case

$$L_u s = L_{s_1 \circ v} s = L_{s_1} L_v s = s_1 (L_v s) = [s_1, L_v s] + (L_v s) s_1.$$

Since $(L_v s) s_1 \in [\mathfrak{A}_S]_1$, the initial step of the induction is done.

b) Let us assume that our assertion is true in the case when the degree of the element $x_1 \in [\mathfrak{A}_S]_1$ is not greater than m. Let x_1 have degree m + 1, so one can assume that $x_1 = ys_1$, where $s_1 \in S$, and the degree of y is equal to m; correspondingly, $u = (ys_1) \circ v$. We have

$$L_u s = L_{(ys_1)} L_v s = (ys_1)(L_v s) = [(ys_1), L_v s] + (L_v s)(ys_1).$$

Further,

$$L_{(L_v s)}(ys_1) = L_{(L_v s)} L_y s_1 = L_{[L_v s, y]} s_1 + L_y L_{(L_v s)} s_1 = L_{[L_v s, y]} s_1 + L_{y \circ (L_v s)} s_1.$$

Obviously, $L_{[L_v s, y]} s_1 \in [\mathfrak{A}_S]_1$. In addition, the element y, having degree less than n, by virtue of the inductive hypothesis of the first induction is the sum of elements of the form $z_1 \circ \ldots \circ z_l$, where $z_i \in [\mathfrak{A}_S]_1$. Here the degree of z_1 does not exceed the degree of y, i.e., m.

Correspondingly, $y \circ (L_v s)$ is the sum of the elements $z_1 \circ \ldots \circ z_l \circ (L_v s) = z_1 \circ w$. Consequently, the element $L_{y \circ}(L_v s) s_1$ falls under the action of the inductive hypothesis of the second induction.

The proof of the proposition is finished.

We denote by $[\mathscr{A}]_1$ any basis of the vector space $[\mathfrak{A}_S]_1$. The elements of the set $[\mathscr{A}]_1$ generate the whole Lie algebra $[\mathfrak{A}_S]$, however they are not independent generators of this Lie algebra. We consider, for example, the case when the set S consists of one element $S = \{s\}$, and the basis $[\mathscr{A}]_1$ is homogeneous (in the sense of degree). In this case, the first "commutator" relation between elements of the set $[\mathscr{A}]_1$ arises in degree 5. In degree 5 such a relation is unique and can be found by direct calculation. In the notation introduced in the table on p. 1655 it can be written as follows:

$$3y_5 s - 5y_6 s + 2y_8 s = 3[y_1, y_6 - y_5] + [y_2, y_3] + [y_2, [y_1, y_2]] - [y_1, [y_1, y_3]].$$

One should note that in components of degrees 1-5 the graded Lie algebra $[\mathfrak{A}_1]$ coincides with the free Lie algebra with generators, let us say y_1, y_2, y_3, y_5, y_6, $y_5 s$, $y_6 s$, $y_7 s$.

The question arises, is the Lie algebra $[\mathfrak{A}_1]$, or more generally, is the Lie algebra $[\mathfrak{A}_S]$, for arbitrary S the free Lie algebra with the corresponding set of generators? We still do not know the answer to this question.

3. Group of Formal Flows of a Graded ch-Algebra. Let $\mathfrak{A}$ be a graded ch-algebra over the field K of characteristic zero. Then $\mathfrak{A} = \sum_{n=1}^{\infty} \mathfrak{A}_n$, where $\mathfrak{A}_n$ is the n-th component of this algebra, n = 1, 2, ... By the completion of the graded algebra $\mathfrak{A}$ is meant the ch-algebra $\bar{\mathfrak{A}} = \prod_{n=1}^{\infty} \mathfrak{A}_n$. The elements of $\bar{\mathfrak{A}}$ can be written in the form of formal series, $\bar{\mathfrak{A}} = \left\{ x = \sum_{n=1}^{\infty} x_n | x_n \in \mathfrak{A}_n,\ n = 1, 2, \ldots \right\}$, and the multiplication of such series is defined as

usual a la Cauchy: if $x=\sum_{n=1}^{\infty} x_n$, $y=\sum_{n=1}^{\infty} y_n$, then $xy=\sum_{n=2}^{\infty}\left(\sum_{i=1}^{n-1} x_i y_{n-i}\right)$.

The algebra $\mathfrak{A}$ is imbedded in the obvious way as a subalgebra of $\bar{\mathfrak{A}}$.

In addition to the ch-algebra $\bar{\mathfrak{A}}$, we shall use for auxiliary constructions the ch-algebra with unit $\bar{\mathfrak{A}}\oplus K$ (cf. p. 1651).

We define a map $W:\bar{\mathfrak{A}}\to\bar{\mathfrak{A}}$ by the formula

$$W(x)=1-e^{-L_x}1=\sum_{m=1}^{\infty}\frac{1}{m!}(-L_x)^{m-1}1=\sum_{m=0}^{\infty}\frac{1}{(m+1)!}(-L_x)^m x.$$

The map W is well-defined. In fact, the first nonzero term of the series $(-L_x)^m x\in\bar{\mathfrak{A}}$ has degree not less than m + i. We write the first terms of the series for W(x) separately

$$W(x)=x-\frac{1}{2}xx+\frac{1}{6}x(xx)-\ldots$$

In the homogeneous component of degree 1, the map W coincides with the identity, so W is invertible (implicit function theorem for formal series). We introduce the notation

$$V(x)=W^{-1}(x).$$

Now we find a functional equation for V, which allows us to calculate V(x) recursively up to higher degrees.

We denote by $\chi(\varepsilon)=\frac{\varepsilon}{1-e^{-\varepsilon}}=1+\frac{\varepsilon}{2}+\sum_{k=0}^{\infty}\frac{B_k}{k!}\varepsilon^k$ the formal series in the variable ε. (B_k is the k-th Bernoulli number.) We have

$$x=W(V(x))=1-e^{-L_{V(x)}}1=(1-e^{-L_{V(x)}})1=\frac{(1-e^{-L_{V(x)}})}{L_{V(x)}}V(x)=1/\chi(L_{V(x)})V(x).$$

Consequently,

$$V(x)=\chi(L_{V(x)})x. \tag{1.5}$$

It is easy to see that V(x) [as well as W(x)] can be represented in the form

$$V(x)=\sum_{i=1}^{\infty}V_i(x),$$

where V_i is a homogeneous (nonassociative) polynomial of degree i in the variable x. Equation (1.5) makes it possible to calculate the polynomial V_n, $\forall n$, with the help of the polynomials V_i with i < n. Here are the first four polynomials:

$$V_1(x)=x,\ V_2(x)=\frac{1}{2}L_{V_1(x)}x=\frac{1}{2}xx,\ V_3(x)=\frac{1}{4}(xx)x+\frac{1}{6}x(xx),$$

$$V_4(x)=\frac{1}{8}((xx)x)x+\frac{1}{12}(x(xx))x+\frac{1}{12}x((xx)x)+\frac{1}{12}(xx)(xx).$$

We define, finally, the map

$$f:\bar{\mathfrak{A}}\times\bar{\mathfrak{A}}\to\bar{\mathfrak{A}}$$

by the formula

$$f(x,y)=e^{-L_{V(y)}}x+y,\ \forall x,\ y\in\bar{\mathfrak{A}}.$$

The space $\bar{\mathfrak{A}}$ with the nonlinear multiplication operation (x, y) ↦ f(x, y) is called the group of formal flows of the ch-algebra $\mathfrak{A}$.

The fact that this is actually a group will be proved a little later; the origin of the name will become clear from Sec. 2.

Example. Let us assume that the ch-algebra $\mathfrak{A}$ is associative. Then the ch-algebra $\bar{\mathfrak{A}}$ is also associative. In this case all the formulas simplify a great deal. For example,

$$W(x)=1-e^{-L_x}1=1-e^{-x}=\sum_{n=1}^{\infty}\frac{(-x)^{n+1}}{n!},$$

$$V(x)=W^{-1}(x)=-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n},$$

$$f(x, y)=e^{-V(y)}x+y=(1-y)x+y=x+y-yx.$$

Below, we shall need the Campbell–Hausdorff series of the Lie algebra $[\mathfrak{A}]$. We recall the definition of this series.

Let $\mathfrak{L}=\sum_{n=1}^{\infty}\mathfrak{L}_n$ be an arbitrary graded Lie algebra over k, $\bar{\mathfrak{L}}=\left\{x=\sum_{n=1}^{\infty}x_n \mid x_n\in\mathfrak{L}_n,\ n=1,2,\dots\right\}$ be its completion. Further, $U(\mathfrak{L})$ is the universal enveloping algebra with the corresponding grading, $\overline{U(\mathfrak{L})}$ is its completion. The (associative) multiplication in $\overline{U(\mathfrak{L})}$ will be denoted by a little circle "$\circ$." Let $x\in\bar{\mathfrak{L}}$, so the element $e^x=\sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{x\circ\dots\circ x}_{n\text{ times}}$ of the algebra $\overline{U(\mathfrak{L})}$ is defined.

It turns out that $\forall x,\ y\in\bar{\mathfrak{L}}$ the element

$$\ln(e^x\circ e^y)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\underbrace{(e^x\circ e^y-1)\circ\dots\circ(e^x\circ e^y-1)}_{n\text{ times}}$$

lies in the Lie algebra $\bar{\mathfrak{L}}\subset\overline{U(\mathfrak{L})}$ and can be represented in the form of a series in the variables x and y, containing only the operations of addition, multiplication by scalars, and commutation (but not the associative multiplication "$\circ$").

This series is also called the Campbell–Hausdorff series (for details cf. [5]). We write $h(x, y)=\ln(e^x\circ e^y)\in\bar{\mathfrak{L}}$. It is easy to see that the operation $(x, y)\mapsto h(x, y)$ gives a group structure in the space $\bar{\mathfrak{L}}$. We call this group the Hausdorff group of the algebra $\mathfrak{L}$.

We turn to the ch-algebra $\bar{\mathfrak{A}}$.

Proposition 13. For any $x, y\in\bar{\mathfrak{A}}$ one has

$$W(h(x, y))=f(W(x), W(y)).$$

From the proposition formulated it follows immediately that the operation $(x, y)\mapsto f(x, y)$ actually defines in $\bar{\mathfrak{A}}$ a group structure, while this group is isomorphic with the Hausdorff group of the Lie algebra $[\mathfrak{A}]$, and the isomorphism is established by the maps W and $V=W^{-1}$.

Proof. The necessary equation is established by a direct calculation. In it we use the fact that h(x, y) can be expressed in terms of commutators of the elements x and y:

$$f(W(x), W(y))=e^{-L_{V(W(y))}}W(x)+W(y)$$

$$=e^{-L_y}(1-e^{-L_x}1)+(1-e^{-L_y}1)=1-e^{-L_y}e^{-L_x}1=1-e^{-h(L_x,L_y)}1=1-e^{-L_{h(x,y)}}1=W(h(x, y)).$$

The assertion of Proposition 1.3 is equivalent, of course, with the equation

$$V(f(x, y))=h(V(x), V(y))\ \forall x,\ y\in\bar{\mathfrak{A}}.$$

As a corollary, we establish the following useful identity:

$$e^{L_{V(f(x,y))}}=e^{L_{V(x)}}e^{L_{V(y)}}. \tag{1.6}$$

Here is its derivation:

$$e^{L_{V(f(x,y))}}=e^{L_{h(V(x),V(y))}}=e^{h(L_{V(x)},L_{V(y)})}=e^{L_{V(x)}}e^{L_{V(y)}}.$$

The unit element in the group of formal flows, just as in the Hausdorff group, is, obviously, the zero of the algebra $\bar{\mathfrak{A}}$. We shall give a formula for calculating the inverse element of a given $x\in\bar{\mathfrak{A}}$ in the group of formal flows. From the equation $f(y, x)=0$, we get

$$y=-e^{L_{V(x)}}x.$$

Example. Here we use the notation from Example c) of Paragraph I. Let x(t), y(t) lie in some graded subalgebra of the algebra $\mathfrak{L}_T$ (for example, be polynomials with values in $\mathfrak{L}$ without free term). It follows from

Proposition 1.3 that

$$h(x(t),\ y(t))=V(f(W(x),\ W(y))(t)=V(1-e^{-L_y}e^{-L_x}1)(t).$$

If in this formula one sets $x(t) = tx_0$, $y(t) = ty_0$, where x_0, y_0 are some elements of $\mathfrak{L}$, then we get the explicit Campbell–Hausdorff formula

$$h(x_0,\ y_0)=V(1-e^{-L_{ty_0}}e^{-L_{tx_0}}1)|_{t=1}.$$

In fact, the right side of the last equation is the "commutator series" of x_0 and y_0.

From the considerations of Sec. 2, it becomes clear that many classification problems of ordinary differential equations reduce to the calculation of the space of cosets of the group of formal flows (of the corresponding ch-algebra) by some subgroup. To conclude this section, we give a formula, very useful in considering such questions.

We want to transform the equation

$$f(x,\ W(z))=y, \text{ where } x,\ y,\ z\in\bar{\mathfrak{A}}.$$

We get

$$e^{-L_z}x+W(z)=y,\ e^{-L_z}x+1-e^{-L_z}1=y,\ e^{-L_z}(x-1)=y-1. \tag{1.7}$$

The map $z\mapsto e^{L_z}$ is a linear representation of the Hausdorff group of the Lie algebra $[\mathfrak{A}]$ into the space $\bar{\mathfrak{A}}\oplus K$. It follows from Proposition 1.3 that some subset $Z\subset\bar{\mathfrak{A}}$ is a subgroup of the Hausdorff group of the Lie algebra $[\mathfrak{A}]$ if and only if $W(Z)$ is a subgroup of the group of formal flows of the ch-algebra $\mathfrak{A}$. Thus, the study of cosets of the group of formal flows by some subgroup is reduced to the study of the orbit space of the linear representation $z\mapsto e^{L_z}$ of the corresponding subgroup of the Hausdorff group on the hyperplane $\bar{\mathfrak{A}}-1$.

2. Nonstationary Vector Fields

1. Preliminary Information. In this paragraph some definitions and formulas used in what follows are collected. Basically this is standard material, of which there is a detailed account in the form we need in [3]. In the first section of the cited paper one can find all the proofs omitted here.

Let M be some manifold of class C^∞, infinitely differentiably imbedded in a d-dimensional real vector space R^d. The algebra of all smooth (i.e., infinitely differentiable) functions on M we denote by Φ; the Cartesian product of d copies of the vector space Φ we denote by Φ^d.

Any linear map A of the vector space Φ into itself corresponds to a linear map of Φ^d into itself, defined by the formula

$$A\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_d\end{pmatrix}=\begin{pmatrix}A\varphi_1\\ \vdots\\ A\varphi_d\end{pmatrix},\quad \varphi_i\in\Phi,\ i=1,\dots,\ d.$$

The map of M into $\mathbf{R}^d$ which is the restriction to M of the identity transformation of the space $\mathbf{R}^d$ we denote by the letter E. Thus, $\forall x=\begin{pmatrix}x_1\\ \vdots\\ x_d\end{pmatrix}\in M\subset\mathbf{R}^d,\quad E(x)=x$. It is clear that $E\in\Phi^d$.

By the symbol T_xM we shall denote the tangent space of the manifold M at the point $x\in M$. Then T_xM is a space of dimension dim M in $\mathbf{R}^d$.

A linear map $X:\Phi\to\Phi$ is called a differentiation of the algebra Φ if

$$X(\varphi_1\varphi_2)=(X\varphi_1)\varphi_2+\varphi_1(X\varphi_2)\quad \forall\varphi_1,\ \varphi_2\in\Phi.$$

The commutator $[X_1, X_2]=X_1\circ X_2-X_2\circ X_1$ of two differentiations X_1 and X_2 is again a differentiation. The operation of commutation defines in the space of all differentiations the structure of a Lie algebra. We denote the Lie algebra of all differentiations of the algebra Φ by the symbol $D(\Phi)$.

Just as in any Lie algebra, to each element $X\in D(\Phi)$ corresponds a linear map ad X, acting on arbitrary $Y\in D(\Phi)$ according to the rule

$$(\mathrm{ad}\,X)Y=[X,Y].$$

If one sets $[\mathrm{ad}\, X_1, \mathrm{ad}\, X_2] \overset{\mathrm{def}}{=} (\mathrm{ad}\, X_1)\circ \mathrm{ad}\, X_2) - (\mathrm{ad}\, X_2 \circ \mathrm{ad}\, X_1) \forall X_1, X_2 \in D(\Phi)$, then from the Jacobi identity follows the equation

$$[\mathrm{ad}\, X_1, \mathrm{ad}\, X_2] = \mathrm{ad}\, [X_1, X_2].$$

A differentiation can be multiplied by elements of the algebra Φ. If $X \in D(\Phi)$ and $\varphi \in \Phi$, then by definition,

$$(\varphi X)\, \varphi_1 = \varphi\, (X \varphi_1) \quad \forall \varphi_1 \in \Phi.$$

It is clear that the map $\varphi X : \Phi \to \Phi$ so defined is a differentiation.

As is known, any differentiation of the algebra Φ is a first order differential operator and acts according to the formula

$$(X\varphi)(x) = \langle d\varphi(x), XE(x) \rangle \quad \forall x \in M, \ \forall \varphi \in \Phi.$$

Here d is the differential of the function φ at the point x, XE is the function from Φ^d obtained by applying X to the function E $(XE(x) \in T_x M, \ \forall x \in M)$, and the brackets denote application of the linear form $d\varphi(x)$ to the vector XE(x).

Differentiations of the algebra Φ are also called smooth vector fields (or simply fields) on the manifold M, and the Lie algebra $D(\Phi)$ is the Lie algebra of vector fields on M.

For any $x \in M$, let $\pi(x) : \mathbf{R}^d \to M$ be an orthogonal projector of the space $\mathbf{R}^d$ onto $T_x M$. To each vector $h \in \mathbf{R}^d$ we make correspond the vector field $\vec{h} \in D(\Phi)$, acting according to the rule

$$\vec{h}\varphi(x) = \langle d\varphi(x), \pi(x) h \rangle .$$

Thus, $\vec{h}E(x) = \pi(x)h$.

In Φ we define a seminorm $\|\cdot\|_{s,M}$ for arbitrary integral $s \geq 0$ and an arbitrary set $U \subset M$, which can also assume infinite values, defining them by the formula

$$\|\varphi\|_{s,M} = \sup_{x \in U} \sum_{\alpha=1}^{s} \sup_{\substack{|h_j|=1 \\ 1 \leq j \leq \alpha}} |\vec{h}_1 \circ \ldots \circ \vec{h}_\alpha \varphi(x)| \quad \forall \varphi \in \Phi.$$

If $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_d \end{pmatrix} \in \Phi_d$, then we set $\|\varphi\|_{s,U} = \max_\alpha \|\varphi_\alpha\|_{s,U}$. Finally, for any vector field $X \in D(\Phi)$, we set $\|X\|_{s,U} = \|XE\|_{s,U}$. For U = M, the indication of the set in the notation for the seminorm will be omitted, $\|\cdot\|_s \overset{\mathrm{def}}{=} \|\cdot\|_{s,M}$.

We introduce the topology in Φ, defined by the family of seminorms $\|\cdot\|_{s,K}$, where $s \geq 0$, K is an arbitrary compactum in $\mathbf{R}^d$. This topology, called the topology of compact convergence with all derivatives, turns Φ into a Frechet space (a complete, metrizable, locally convex space), and in what follows, we shall always consider this topology in Φ.

By $\mathscr{L}(\Phi)$ we denote the associative algebra of continuous linear operators on Φ into itself. The product of two linear maps A_1 and A_2 is their composition $A_1 \circ A_2$. It is easy to prove that any vector field X from $D(\Phi)$ is a continuous transformation of the space Φ, i.e., belongs to $\mathscr{L}(\Phi)$. More exactly, one has

$$\|X\varphi\|_{s,K} \leq C(s, K)\, \|X\|_{s,K}\, \|\varphi\|_{s+1,K} \quad \forall \varphi \in \Phi.$$

With each smooth map P of the manifold M into itself we associate a linear transformation $\hat{P}$ of the space Φ, defining it by the formula

$$\hat{P}\varphi = \varphi \circ P,$$

where $\varphi \circ P$ is the composition of the map P and φ, $(\varphi \circ P)(x) = \varphi(P(x))$.

One can show that $\hat{P}$ is a continuous transformation, i.e., that $\hat{P} \in \mathscr{L}(\Phi)$. If $Q = P_1 \circ P_2$ is the composition of the mappings P_2 and P_1, then obviously, $\hat{Q} = \hat{P}_2 \circ \hat{P}_1$.

In what follows, all smooth maps P which appear will, as a rule, be diffeomorphisms of the manifold M. The corresponding linear transformations $\hat{P} \in \mathscr{L}(\Phi)$ will also be called diffeomorphisms. The collection of all diffeomorphisms forms a group with respect to composition. Here one has the identity

$$(\hat{P})^{-1} = (\widehat{P^{-1}}).$$

If $\hat{P}$ is a diffeomorphism, then by direct calculation one verifies that for any vector field $X \in D(\Phi)$, the composition $\hat{P} \circ X \circ \hat{P}^{-1}$ is also a field. We consider the map $X \mapsto \hat{P} \circ \hat{X} \circ \hat{P}^{-1}$ as a linear transformation of the space $D(\Phi)$ and we denote it by $\operatorname{Ad} \hat{P}$,

$$(\operatorname{Ad} \hat{P}) X \overset{\text{def}}{=} \hat{P} \circ X \circ \hat{P}^{-1} \in D(\Phi) \ \forall X.$$

It is easy to see that $(\operatorname{As} \hat{P}_1) \circ (\operatorname{Ad} \hat{P}_2) = \operatorname{Ad}(\hat{P}_1 \circ \hat{P}_2)$ for any diffeomorphisms $\hat{P}_1$, $\hat{P}_2$.

Later, we shall have to do with families φ_t, $t \in \mathbf{R}$, of elements of Φ, to which, in the standard way, one can carry over the basic constructions of analysis, if one uses the topology in Φ. We note here only the most necessary ones.

Continuity and differentiability in t of the family φ_t do not require special definitions, since Φ is a topological vector space. A family φ_t, $t \in \mathbf{R}$ will be called measurable if $\forall x \in M$ the scalar function $t \mapsto \varphi_t(x)$ is measurable, and any measurable family will be called locally integrable if for any given t_1, t_2, $s \geq 0$ and compact $K \subset M$

$$\int_{t_1}^{t_2} \| \varphi_\tau \|_{s,K} d\tau < \infty.$$

By the integral of a locally integrable family φ_t, $t \in \mathbf{R}$, between given limits t_1, t_2, we mean the function

$$x \mapsto \int_{t_1}^{t_2} \varphi_\tau(x)\, d\tau, \quad x \in M.$$

It is easy to prove that the function so defined belongs to Φ, while

$$\left\| \int_{t_1}^{t_2} \varphi_\tau d\tau \right\|_{s,K} \leqslant \int_{t_1}^{t_2} \| \varphi_\tau \|_{s,K} d\tau.$$

We call the family φ_t, $t \in \mathbf{R}$ absolutely continuous if there exists a locally integrable family ψ_t such that $\varphi_t = \varphi_{t_0} + \int_{t_0}^{t} \psi_t d\tau$. Using the fact that in Φ there exists a countable everywhere dense set of elements, one can prove, just as for scalar functions, that for almost all t

$$\frac{d}{dt} \varphi_t = \frac{d}{dt} \int_{t_0}^{t} \psi_\tau d\tau = \psi_t.$$

Now we proceed to the consideration of families of linear transformations A_t, $t \in \mathbf{R}$, from $\mathscr{L}(\Phi)$, where all the concepts of analysis, defined above for families φ_t, automatically carry over to families of linear transformations, if one defines the corresponding concepts for A_t in the "weak" sense.

We define measurability, continuity, differentiability, local integrability, absolute continuity of a family A_t, $t \in \mathbf{R}$, by requiring that $\forall \varphi \in \Phi$ the family $A_t \varphi$ should have the corresponding property.

By the derivative of a family A_t which is differentiable at the point t_0 is meant the linear transformation acting according to the formula

$$A'_{t_0} \varphi = \lim_{\delta t \to 0} \frac{1}{\delta t} (A_{t_0 + \delta t} - A_{t_0}) \varphi.$$

Correspondingly, the integral of a locally summable family A_t between the limits t_1 and t_2 is the linear transformation defined by the formula

$$\int_{t_1}^{t_2} A_\tau d\tau \varphi = \int_{t_1}^{t_2} A_\tau \varphi d\tau.$$

Using the Banach–Steinhaus theorem, one can prove that the transformations A'_{t_0} and $\int_{t_1}^{t_2} A_\tau d\tau$ are continuous, i.e., belong to $\mathscr{L}(\Phi)$.

If the family A_t, $t \in \mathbf{R}$ is absolutely continuous, then there exists a locally integrable family $\tilde{A}_t$ such that $A_t = A_{t_0} + \int_{t_0}^{t} \tilde{A}_\tau d\tau$. From the analogous proposition for families φ_t it follows immediately that $(d/dt)A_t = \tilde{A}_t$ for almost all t.

It is easy to see that if the family B_t is locally integrable and the family A_t is continuous, then the family $A_t \circ B_t$ is locally integrable.

Finally, if A_t and B_t are absolutely continuous families, then the family $A_t \circ B_t$ is absolutely continuous also and for it one has the formula for differentiating the product

$$\frac{d}{dt}(A_t \circ B_t) = \left(\frac{d}{dt} A_t\right) \circ B_t + A_t \circ \left(\frac{d}{dt} B_t\right).$$

We shall have to do, basically, with two classes of families of operators from $\mathscr{L}(\Phi)$, nonstationary fields and flows.

By a nonstationary field on M or simply a field, we mean an arbitrary locally integrable family X_t, $t \in \mathbf{R}$ of vector fields on M, $X_t \in D(\Phi)$ $\forall t \in \mathrm{R}$. A nonstationary field is called bounded if

$$\int_{t_1}^{t_2} \| X_\tau \|_s d\tau < \infty \quad \forall s \geqslant 0, \quad \forall t_1, t_2 \in \mathrm{R}.$$

One should distinguish a nonstationary field, which is a family of vector fields from $D(\Phi)$, from the elements of this family. Sometimes, when it is important to emphasize this distinction, we denote the nonstationary field X_t, $t \in \mathrm{R}$ by one letter X, omitting the index t, similarly to the way in denoting a function one omits the argument. At the same time it is often convenient to preserve the symbol X_t also for denoting the entire family of vector fields. This is done in those cases when it is clear from the context whether one is dealing with the entire family $\{X_t \mid t \in \mathbf{R}\}$ or with the one element of this family corresponding to the given moment t.

By a flow on M we mean an arbitrary absolutely continuous family of diffeomorphisms $\hat{P}_t$, $t \in \mathbf{R}$, satisfying the condition $\hat{P}_0 = \mathrm{Id}$. We shall also call the family of diffeomorphisms P_t corresponding to the flow $\hat{P}_t$ a flow.

Let X_t be some nonstationary field. We consider the linear differential equation

$$\frac{d}{dt} A_t = A_t \circ X_t \tag{2.1}$$

with the initial condition

$$A_0 = \mathrm{Id} \tag{2.2}$$

with respect to the unknown family A_t of linear transformations from $\mathscr{L}(\Phi)$. By a solution of this equation with the given initial condition is meant any absolutely continuous family A_t, $t \in \mathbf{R}$, satisfying (2.1) for almost all $t \in \mathbf{R}$ and the condition (2.2). The absolute continuity of the solution sought guarantees the equivalence of equations (2.1)-(2.2) with the integral equation

$$A_t = \mathrm{Id} + \int_0^t A_\tau \circ X_\tau d\tau. \tag{2.3}$$

The linear differential equation

$$\frac{d}{dt} B_t = -X_t \circ B_t$$

will be called adjoint with (2.1). To it corresponds the integral equation

$$B_t = \mathrm{Id} - \int_0^t X_\tau \circ B_\tau d\tau. \tag{2.4}$$

We call an absolutely continuous family invertible if $\forall t \in \mathrm{R}$ the linear transformation A_t has an inverse $A_t^{-1} \in \mathscr{L}(\Phi)$, and the family A_t^{-1} is also absolutely continuous. An arbitrary flow $\hat{P}_t$ is invertible because using the implicit function theorem one can show that the family $\hat{P}_t^{-1}$ depends absolutely continuously on t.

Let us assume that some flow $\hat{P}_t$ is a solution of (2.1). Then the corresponding flow P_t in the manifold M is defined by the ordinary differential equation

$$\frac{dx}{dt} = X_t E(x), \quad x \in M \subset \mathbf{R}^d, \tag{2.5}$$

because

$$\frac{dP_t}{dt}=\frac{d}{dt}\hat{P}_tE=\hat{P}_t\circ X_tE=(X_tE)\circ P_t.$$

Conversely, if P_t is the flow in M, defined by the ordinary differential equation (2.5), then the corresponding family $\hat{P}_t$ of elements of $\mathscr{L}(\Phi)$ is a solution of (2.1).

If B_t is an arbitrary solution of (2.4), then $\forall\varphi\in\Phi$ the function $(t, x)\mapsto B_t\varphi(x)=\omega_t(x)$ satisfies the linear homogeneous first order partial differential equation

$$\frac{\partial\omega_t}{\partial t}+X_t\omega_t=0,\quad \omega_0(x)=\varphi(x).$$

Proposition 2.1. Let X_t be a bounded nonstationary field. Then each of the equations (2.3) and (2.4) has a unique solution, while these solutions are mutually inverse flows.

Proof. First, we note that any solution A_t of (2.3) is a left inverse for any solution B_t of (2.4). In fact,

$$\frac{d}{dt}(A_t\circ B_t)=\left(\frac{d}{dt}A_t\right)\circ B_t+A_t\circ\left(\frac{d}{dt}B_t\right)=A_t\circ X_t\circ B_t-A_t\circ X_t\circ B_t=0.$$

Consequently, $A_t\circ B_t\equiv A_0\circ B_0=\mathrm{Id}$.

In addition, if A_t is an invertible solution of (2.3), then the family A_t^{-1} is a solution of (2.4), because

$$0=\frac{d}{dt}(A_t\circ A_t^{-1})=A_t\circ X_t\circ A_t^{-1}+A_t\circ\frac{d}{dt}A_t^{-1},$$
$$\frac{d}{dt}A_t^{-1}=-X_t\circ A_t^{-1}.$$

Thus, taking into account the remarks made before the formulation of Proposition 2.1, this proposition becomes a consequence of the theorem of existence of solutions for ordinary differential equations.

2. Chronological Exponential and Chronological Logarithm. Definition. Let X_t be a bounded nonstationary field. The flow $\hat{P}_t$, which is the unique solution of the differential equation

$$\frac{d}{dt}\hat{P}_t=\hat{P}_t\circ X_t,\quad \hat{P}_0=\mathrm{Id},$$

is called the right chronological exponential of the absolutely continuous nonstationary field $\int_0^t X_\tau d\tau$ and we introduce the special notation

$$\hat{P}_t=\overrightarrow{\exp}\int_0^t X_\tau d\tau.$$

Correspondingly, we call the absolutely continuous field $\int_0^t X_\tau d\tau$ the right chronological logarithm of the flow $\hat{P}_t$ and we write

$$\int_0^t X_\tau d\tau=\overrightarrow{\ln}\hat{P}_t$$

(strictly speaking, one should write

$$\hat{P}_t=\overrightarrow{\exp}\left\{\int_0^\tau X_\theta d\theta\,|\,0\leqslant\tau\leqslant t\right\}\ \text{and}\ \int_0^t X_\tau d\tau=\ln\{\hat{P}_\tau\,|\,0\leqslant\tau\leqslant t\};$$

however, we shall use the shortened notation). Thus $\overrightarrow{\ln}\,\overrightarrow{\exp}\int_0^t X_\tau d\tau=\int_0^t X_\tau d\tau,\ \overrightarrow{\exp}\,\overrightarrow{\ln}\hat{P}_t=\hat{P}_t$.

Analogously, a flow Q_t, satisfying the equation

$$\frac{d}{dt}\hat{Q}_t=X_t\circ\hat{Q}_t,\quad \hat{Q}_0=\mathrm{Id},$$

is called the left chronological exponential of $\int_0^t X_\tau d\tau$, and the field $\int_0^t X_\tau d\tau$ is called the left chronological logarithm of the flow $\hat{Q}_t$. Here one uses the notation

$$\hat{Q}_t = \overleftarrow{\exp}\int_0^t X_\tau d\tau, \quad \int_0^t X_\tau d\tau = \overleftarrow{\ln}\,\hat{Q}_t$$

(the direction of the arrow is from right to left).

From Proposition (2.1) follows the formula

$$\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau\right)^{-1} = \overleftarrow{\exp}\int_0^t -X_\tau d\tau. \tag{2.6}$$

As we remarked already, the operation of composition "∘" turns the collection of all diffeomorphisms of the manifold M into a group. Correspondingly, the operation which associates with the pair of flows $\hat{P}_t$, $\hat{Q}_t$ the flow $P_t \circ Q_t$, $t \in \mathbf{R}$, defines a group structure on the collection of all flows on M. We call the group so obtained the group of flows (not formal!) on M.

Let X_t, Y_t be two bounded nonstationary fields. Our next goal is to find the absolutely continuous field

$$\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right).$$

Temporarily, we denote the flow $\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau$ by $\hat{P}_t$. We have

$$\frac{d}{dt}\hat{P}_t = \frac{d}{dt}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right) = \overrightarrow{\exp}\int_0^t X_\tau d\tau \circ X_t \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau$$

$$+ \hat{P}_t \circ Y_t = \hat{P}_t \circ \left(\left(\overrightarrow{\exp}\int_0^t Y_\tau d\tau\right)^{-1} \circ X_t \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau + Y_t\right)$$

or, using the symbol Ad (cf. p. 1661) and (2.6),

$$\frac{d}{dt}\hat{P}_t = \hat{P}_t \circ \left(\left(\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau\right)X_t + Y_t\right).$$

Consequently, $\overrightarrow{\ln}\hat{P}_t = \int_0^t \left(\left(\mathrm{Ad}\,\overrightarrow{\exp}\int_0^\tau -Y_\theta d\theta\right)X_\tau + Y_\tau\right)d\tau$. In the formula obtained $\left(\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau\right)$ is a family depending on $t \in \mathbf{R}$ of linear transformations of the space $D(\Phi)$. Let $Z \in D(\Phi)$. Differentiation of $\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau Z$ with respect to t gives the equation

$$\frac{d}{dt}\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau Z = \frac{d}{dt}\left(\overleftarrow{\exp}\int_0^t -Y_\tau d\tau \circ Z \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right)$$

$$= -Y_t \circ \left(\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau\right)Z + \left(\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau\right)Z \circ Y_t = -\mathrm{ad}\,Y_t \circ \left(\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau\right)Z.$$

Since Z is arbitrary, this equation can be considered, at least formally, as a linear equation for $\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau$:

$$\frac{d}{dt}\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau = -\mathrm{ad}\,Y_t \circ \mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau, \quad \mathrm{Ad}\,\mathrm{Id} = \mathrm{Id}. \tag{2.7}$$

Taking account of the equation written, by analogy with the chronological exponentials of vector fields, we use to denote $\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau$ the symbol "left chronological exponential of $-\mathrm{ad}\,Y_\tau$":

$$\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau = \overleftarrow{\exp}\int_0^t -\mathrm{ad}\,Y_\tau d\tau.$$

In exactly the same way for a family $\mathrm{Ad}\,\overrightarrow{\exp}\int_0^t Y_\tau d\tau = \left(\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t -Y_\tau d\tau\right)^{-1}$ and arbitrary $Z \in D(\Phi)$ one has

$$\frac{d}{dt}\left(\mathrm{Ad}\,\overrightarrow{\exp}\int_0^t Y_\tau d\tau\right)Z = \left(\mathrm{Ad}\,\overrightarrow{\exp}\int_0^t Y_\tau d\tau\right)\circ(\mathrm{ad}\,Y_t)Z.$$

Hence, we introduce the notation

$$\mathrm{Ad}\,\overrightarrow{\exp}\int_0^t Y_\tau d\tau = \overrightarrow{\exp}\int_0^t \mathrm{ad}\,Y_\tau d\tau.$$

Then one has the analogue of (2.6)

$$\left(\overrightarrow{\exp}\int_0^t \mathrm{ad}\,Y_\tau d\tau\right)^{-1} = \overleftarrow{\exp}\int_0^t -\mathrm{ad}\,Y_\tau d\tau. \tag{2.8}$$

Taking account of the notation just introduced, the expression for the chronological logarithm of the product of two flows assumes the form

$$\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau\circ\overrightarrow{\exp}\int_0^t Y_\tau d\tau\right) = \int_0^t\left(\overrightarrow{\exp}\int_0^\tau -\mathrm{ad}\,Y_\theta d\theta X_\tau + Y_\tau\right)d\tau. \tag{2.9}$$

The use of the symbols for the chronological exponentials of $\pm\mathrm{ad}\,Y$ serves not only for a formal notation for the operator Ad from the corresponding flows. Actually, the family $\mathrm{Ad}\,\overrightarrow{\exp}\int_0^t Y_\tau d\tau$ (as well as $\mathrm{Ad}\,\overleftarrow{\exp}\int_0^t Y_\tau d\tau$ also) can be reestablished, at least asymptotically, from the family $\mathrm{ad}\,Y_t$ of transformations of $D(\Phi)$.

In fact, we set $F_t = \mathrm{Ad}\,\overrightarrow{\exp}\int_0^t Y_\tau d\tau$. One has the equation

$$F_t Z = Z + \int_0^t F_\tau\circ\mathrm{ad}\,Y_\tau Z d\tau, \quad \forall Z \in D(\Phi).$$

Substituting in this equation in place of $F_\tau\circ\mathrm{ad}\,Y_\tau Z$ the expression $\mathrm{ad}\,Y_\tau\circ Z + \int_0^\tau F_\theta\circ\mathrm{ad}\,Y_\theta\circ\mathrm{ad}\,Y_\tau Z d\theta$, we get

$$F_t Z = Z + \int_0^t Y_\tau Z d\tau + \int_0^t d\tau_1\int_0^{\tau_1} d\tau_2\,(F_{\tau_2}\circ\mathrm{ad}\,Y_{\tau_2}\circ\mathrm{ad}\,Y_{\tau_1} Z).$$

Continuing to act along the same lines, we arrive at the equation

$$F_t Z = Z + \sum_{\alpha=1}^{m-1}\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2\ldots\int_0^{\tau_{\alpha-1}} d\tau_\alpha\,(\mathrm{ad}\,Y_{\tau_\alpha}\circ\ldots\circ\mathrm{ad}\,Y_{\tau_1} Z) + R_m,$$

where $R_m = \int_0^t d\tau_1\ldots\int_0^{\tau_{m-1}} d\tau_m\,(F_{\tau_m}\circ\mathrm{ad}\,Y_{\tau_m}\circ\ldots\circ\mathrm{ad}\,Y_{\tau_1} Z)$. Analogously, if one sets $G_t = \mathrm{Ad}\,\overleftarrow{\exp}\int_0^t Y_\tau d\tau$, then

$$G_t Z = Z + \sum_{\alpha=1}^{m-1}\int_0^t d\tau_1\ldots\int_0^{\tau_{\alpha-1}} d\tau_\alpha\,(\mathrm{ad}\,Y_{\tau_1}\circ\ldots\circ\mathrm{ad}\,Y_{\tau_\alpha} Z) + S_m,$$

where $S_m = \int_0^t d\tau_1\ldots\int_0^{\tau_{m-1}} d\tau_m\,(F_{\tau_m}\circ\mathrm{ad}\,Y_{\tau_1}\circ\ldots\circ\mathrm{ad}\,Y_{\tau_m} Z)$.

Proposition 2.2. Let us assume that $\int_0^t \|Y_\tau\|_0 d\tau < 1$. Then for any integral $m, s > 0$, compactum $K \subset M$, and arbitrary $Z \in D(\Phi)$ one has the estimates

$$\left.\begin{array}{l}\left\| \mathrm{Ad}\,\overleftarrow{\exp}\int_0^t Y_\tau d\tau Z - Z - \sum_{\alpha=1}^{m-1}\int_0^t d\tau_1 \ldots \right. \\ \left. \ldots \int_0^{\tau_{\alpha-1}} d\tau_\alpha (\mathrm{ad}\,Y_{\tau_1}\circ\ldots\circ\mathrm{ad}\,Y_{\tau_\alpha} Z)\right\|_{s,K} \\ \left\| \mathrm{Ad}\,\overrightarrow{\exp}\int_0^t Y_\tau d\tau Z - Z - \sum_{\alpha=1}^{m-1}\int_0^t d\tau_1 \ldots \right. \\ \left. \ldots \int_0^{\tau_{\alpha-1}} d\tau_\alpha (\mathrm{ad}\,Y_{\tau_\alpha}\circ\ldots\circ\mathrm{ad}\,Y_{\tau_1} Z)\right\|_{s,K}\end{array}\right\} \leqslant C_1 e^{C_2\int_0^t \|X_\tau\|_{s+1} d\tau} \left(\int_0^t \| X_\tau \|_{s+m,U} d\tau\right)^m \| Z \|_{s+m,U},$$

where C_1, C_2 depend only on s, m, and K, and U is the neighborhood of radius 2 of the compactum K.

For the proof of this proposition, cf. [3].

Remark. If M is a real analytic manifold, and Y_t, Z are bounded analytic fields, then the series

$$\sum_{\alpha=1}^{\infty}\int_0^t d\tau_1 \ldots \int_0^{\tau_{\alpha-1}} d\tau_\alpha (\mathrm{ad}\,Y_{\tau_\alpha}\circ\ldots\circ\mathrm{ad}\,Y_{\tau_1} Z) E(x)$$

converges under specific conditions, and its sum is $\left(\mathrm{Ad}\,\overrightarrow{\exp}\int_0^t Y_\tau d\tau Z\right) E(x)$.

Thus, to each bounded nonstationary field X_t corresponds the flow $\overrightarrow{\exp}\int_0^t X_\tau d\tau$, while one has a formula allowing one to calculate the field to which corresponds the composition of the given flows. Here there is an analogy with Lie theory. In Lie theory, to each element x of a given Lie algebra $\mathfrak{L}$ corresponds an element e^x of the Lie group.

In addition there exists a (Campbell–Hausdorff) formula allowing one for any $x, y \in \mathfrak{L}$, sufficiently close to zero, to calculate $\ln(e^x e^y)$ with the help of only operations of the Lie algebra, i.e., addition, multiplication by a scalar, and commutation. As a result, the study of a Lie group basically reduces to the study of its Lie algebra, a linear object.

In our case, to calculate (even if only asymptotically) the nonstationary field $\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right)$ in terms of $\int_0^t X_\tau d\tau$ and $\int_0^t Y_\tau d\tau$, one uses, in addition to the operations of addition and commutation, also differentiation and integration with respect to t. Thus, the operations of the Lie algebra turn out to be insufficient. Now we describe the chronological algebra, which in this nonstationary situation plays the same role as the Lie algebra of the given group in the classical Lie theory.

In Example c) of Paragraph 1 of the first section, to each finite-dimensional Lie algebra there corresponds a certain chronological algebra. We apply an analogous construction to the infinite-dimensional Lie algebra $D(\Phi)$.

Let $D(\Phi)_T$ be the linear space of all absolutely continuous nonstationary fields X_t, $t \in \mathbf{R}$, satisfying the condition $X_0 = 0$. In the space $D(\Phi)_T$ we define a multiplication operation, turning this space into a ch-algebra. This operation is denoted by "*" and is defined by the formula

$$(X*Y)_t = \int_0^t \left[X_\tau, \frac{d}{d\tau} Y_\tau\right] d\tau \quad \forall X, Y \in D(\Phi)_T.$$

The chronological algebra $D(\Phi)_T$ constructed in this way we call the algebra of nonstationary vector fields. It is easy to see that

$$(X*Y)_t - (Y*X)_t = [X_t, Y_t] \quad \forall X, Y \in D(\Phi)_T.$$

We introduce in addition a commutative associative algebra Φ_T. The algebra Φ_T consists of those measurable families of functions φ_t, $\varphi_t \in \Phi$ $\forall t \in \mathbf{R}$, such that the scalar function $\|\varphi_t\|_{s,K}$ is bounded on each finite segment in $\mathbf{R}$, for any $s \geq 0$ and arbitrary compactum $K \subset M$.

The operation of multiplication in the algebra Φ_T is the usual pointwise multiplication of functions.

We define an action of the algebra Φ_T on the space of nonstationary fields $D(\Phi)_T$ by the formula

$$(\varphi\cdot X)_t=\int_0^t \varphi_\tau \frac{d}{d\tau}X_\tau d\tau,\ \forall\varphi\in\Phi_T,\ X\in D(\Phi)_T.$$

Such an action turns the space $D(\Phi)_T$ into a module over the algebra Φ_T, and the identity

$$(\varphi\psi)\cdot X=\varphi\cdot(\psi\cdot X)\ \forall\varphi,\ \psi\in\Phi_T,\ \forall X\in D(\Phi)_T \tag{2.10}$$

is verified directly.

Further, the Lie algebra $D(\Phi)$ consists of differentiations of the algebra Φ. This action of $D(\Phi)$ on the algebra Φ extends in an obvious way to an action of $D(\Phi)_T$ on Φ_T,

$$(X\varphi)_t=X_t\varphi_t,\ \forall X\in D(\Phi)_T,\ \forall\varphi\in\Phi_T.$$

Here an arbitrary nonstationary field $X \in D(\Phi)_T$ becomes a differentiation of the algebra Φ_T,

$$X(\varphi\psi)=(X\varphi)\psi+\varphi(X\psi),\ \forall\varphi,\ \psi\in\Phi_T. \tag{2.11}$$

The actions of the algebra Φ_T on $D(\Phi)_T$ and conversely of $D(\Phi)_T$ on the algebra Φ_T just described are connected by the following relation:

$$X*(\varphi\cdot Y)=\varphi\cdot(X*Y)+(X\varphi)\cdot Y,\quad \forall X,\ Y\in D(\Phi)_T,\ \varphi\in\Phi_T. \tag{2.12}$$

Equation (2.12) can be verified by direct calculation:

$$(X*(\varphi\cdot Y))_t=\int_0^t\left[X_\tau,\ \varphi_\tau\frac{d}{d\tau}Y_\tau\right]d\tau=\int_0^t\varphi_\tau\left[X_\tau,\ \frac{d}{d\tau}Y_\tau\right]d\tau+\int_0^t(X_\tau\varphi_\tau)\frac{d}{d\tau}Y_\tau d\tau=(\varphi\cdot(X*Y))_t+((X\varphi)\cdot Y)_t.$$

Now we are in the situation to define the structure of a chronological algebra in the space $D(\Phi)_T\oplus\Phi_T$, extending correspondingly the operation "*" from the ch-algebra $D(\Phi)_T$ to the whole space $D(\Phi)_T\oplus\Phi_T$.

Namely, we set

$$\varphi*\psi=\varphi\psi,\quad \varphi*X=\varphi\cdot X,$$
$$X*\varphi=\varphi\cdot X+X\varphi,\quad \forall\varphi,\ \psi\in\Phi_T,\ X\in D(\Phi)_T.$$

It follows easily from (2.10), (2.11), (2.12) that the operation "*" so defined actually turns $D(\Phi)_T\oplus\Phi_T$ into a ch-algebra. We denote this algebra by the symbol $\mathcal{F}_T$.

All the operations noted above with vector fields and functions can be expressed in terms of the operation "*" in $\mathcal{F}_T$. For example, $\forall X\in D(\Phi)_T,\ \varphi\in\Phi_T,\ X\varphi=X*\varphi-\varphi*X$.

The ch-algebra $\mathcal{F}_T$ has a unit [in contrast with the ch-algebra $D(\Phi)_T$]. The unit is the function from Φ_T identically equal to one.

Further, let X_t, $Y_t \in D(\Phi)_T$ be such that $(d/dt)X_t$ and $(d/dt)Y_t$ are bounded nonstationary fields. Then, obviously, $(d/dt)(X*Y)_t$ is also a bounded nonstationary field. Consequently, the subspace of $D(\Phi)_T$ consisting of fields, the derivative with respect to t of which is a bounded field, is a ch-subalgebra of $D(\Phi)_T$. We denote this subalgebra by $BD(\Phi)_T$.

We consider, finally, the ch-algebra $BD(\Phi)_T[\varepsilon]$, consisting of polynomials in the variable ε with coefficients in $BD(\Phi)_T$. The elements of the algebra $BD(\Phi)_T[\varepsilon]$ are expressions of the form $\sum_{i=1}^{m}X_i\varepsilon^i$, where $X_i \in BD(\Phi)_T$ (we consider only polynomials with zero free term). One multiplies polynomials in the usual way:

$$\left(\sum_{i=1}^{m_1}X_i\varepsilon^i\right)*\left(\sum_{j=1}^{m_2}Y_j\varepsilon^j\right)=\sum_{k=2}^{m_1+m_2}\left(\sum_{i+j=k}X_i*Y_j\right)\varepsilon^k.$$

The algebra $BD(\Phi)_T[\varepsilon]$ has the obvious grading, where the component of degree m is the subspace $\{X\varepsilon^m | X \in BD(\Phi)_T[\varepsilon]\}$.

The completion of the graded ch-algebra $BD(\Phi)_T[\varepsilon]$ is the algebra consisting of formal series of the form $\sum_{i=1}^{\infty}X_i\varepsilon^i$, $X_i \in BD(\Phi)_T$, which are multiplied a la Cauchy:

$$\left(\sum_{i=1}^{\infty} X_i\varepsilon^i\right)*\left(\sum_{j=1}^{\infty} Y_j\varepsilon^j\right)=\sum_{k=2}^{\infty}\left(\sum_{i+j=k} X_i * Y_j\right)\varepsilon^k.$$

To denote this completion, we shall use the symbol $BD(\Phi)_T[[\varepsilon]]$.

We have already described the chronological algebras we need and now we can return to the consideration of the formula expressing the chronological logarithm of the product of two flows in terms of the chronological logarithms of the factors. First, from the equation

$$\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right)=\int_0^t\left(\operatorname{Ad}\overrightarrow{\exp}\int_0^\tau Y_\theta d\theta\right)^{-1} X_\tau + Y_\tau d\tau$$

it follows easily that if the nonstationary fields $\int_0^t X_\tau d\tau$, $\int_0^t Y_\tau d\tau$ belong to $BD(\Phi)_T$, then $\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right) \in BD(\Phi)_T$ also. Further, one has

$$\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau\right)^{-1} = -\int_0^t \operatorname{Ad}\overrightarrow{\exp}\int_0^\tau X_\theta d\theta X_\tau d\tau. \tag{2.13}$$

In fact,

$$\overrightarrow{\exp}\int_0^t\left(-\operatorname{Ad}\overrightarrow{\exp}\int_0^\tau X_\theta d\theta\right)d\tau \circ \overrightarrow{\exp}\int_0^t X_\tau d\tau$$

$$=\overrightarrow{\exp}\int_0^t\left(\left(\operatorname{Ad}\overrightarrow{\exp}\int_0^\tau X_\theta d\theta\right)^{-1}\left(-\operatorname{Ad}\overrightarrow{\exp}\int_0^\tau X_\theta d\theta\right)X_\tau + X_\tau\right)d\tau = \overrightarrow{\exp}\int_0^t(-X_\tau + X_\tau)d\tau = \mathrm{Id}.$$

From (2.13) we get that $\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau\right)^{-1}$ lies in $BD(\Phi)_T$, provided $\left(\int_0^t X_\tau d\tau\right) \in BD(\Phi)_T$.

Consequently, the flows whose right chronological logarithms lie in $BD(\Phi)_T$ form a subgroup of the group of all flows. Correspondingly, the map

$$\left(\int_0^t X_\tau d\tau, \int_0^t Y_\tau d\tau\right) \mapsto \overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right) \tag{2.14}$$

defines a group operation in the space $BD(\Phi)_T$.

Let $\left(\int_0^t X_\tau d\tau\right) \in BD(\Phi)_T$. We consider the families of nonstationary fields depending on the variable $\varepsilon \in \mathbf{R}$, $\varepsilon\int_0^t X_\tau d\tau$ and $\varepsilon\int_0^t Y_\tau d\tau$. It follows from Proposition 2.2 that the MacLaurin series with respect to ε of the family of fields

$$\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t \varepsilon X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t \varepsilon Y_\tau d\tau\right)=\int_0^t\left(\overleftarrow{\exp}\int_0^\tau -\operatorname{ad}\varepsilon Y_\theta d\theta \varepsilon X_\tau + \varepsilon Y_\tau\right)d\tau$$

has the form

$$\varepsilon\int_0^t\left(X_\tau + Y_\tau + \sum_{m=1}^{\infty}(-\varepsilon)^m\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{m-1}} d\tau_m(\operatorname{ad} Y_{\tau_1}\circ\ldots\circ\operatorname{ad} Y_{\tau_m} X_\tau)\right)d\tau. \tag{2.15}$$

The formal series (2.15) belongs, of course, to the space $BD(\Phi)_T[[\varepsilon]]$. We denote this series by $l\left(\varepsilon\int_0^t X_\tau d\tau, \varepsilon\int_0^t Y_\tau d\tau\right)$. Let $\sum_{i=1}^{\infty}\varepsilon^i X_i = X(\varepsilon)$ be an arbitrary series from $BD(\Phi)_T$. If in $l(\varepsilon X, \varepsilon Y)$ one substitutes for $\varepsilon\int_0^t X_\tau d\tau$ or $\varepsilon\int_0^t Y_\tau d\tau$ the series $X(\varepsilon)=\varepsilon\left(\sum_{i=0}^{\infty}\varepsilon^i X_{i+1}\right)$, then again we get a formal series from $BD(\Phi)_T[[\varepsilon]]$. Thus, we get a map $l(\cdot,\cdot)$: $BD(\Phi)_T[[\varepsilon]] \times BD(\Phi)_T[[\varepsilon]] \to BD(\Phi)_T[[\varepsilon]]$. This map is the "asymptotic decomposition" of the group operation (2.14).

It is clear that it gives the group structure in $BD(\Phi)_T[[\varepsilon]]$. In particular,

$$l(X(\varepsilon), l(Y(\varepsilon), Z(\varepsilon))) = l(l(X(\varepsilon), Y(\varepsilon)), Z(\varepsilon)),$$

$$\forall X(\varepsilon),\ Y(\varepsilon),\ Z(\varepsilon) \in BD(\Phi)_T[[\varepsilon]].$$

We recall now that $BD(\Phi)_T[[\varepsilon]]$ is a ch-algebra, which is the completion of some graded ch-algebra. According to Paragraph 3 of the first section, there exists a group operation

$$f(\cdot,\cdot): BD(\Phi)_T[[\varepsilon]] \times BD(\Phi)_T[[\varepsilon]] \to BD(\Phi)_T[[\varepsilon]],$$

defining the group of formal flows of the ch-algebra $BD(\Phi)_T[[\varepsilon]]$. Here $f(X(\varepsilon), Y(\varepsilon))$ can be expressed in terms of $X(\varepsilon)$ and $Y(\varepsilon)$ only with the help of the ch-algebra operations, i.e., in our case, the operations of addition, multiplication by a scalar, and "$*$."

Proposition 2.3. The mappings $l(\cdot,\cdot)$ and $f(\cdot,\cdot)$ coincide, i.e., $\forall X(\varepsilon), Y(\varepsilon) \in BD(\Phi)[[\varepsilon]]$ $l(X(\varepsilon), Y(\varepsilon)) = f(X(\varepsilon), Y(\varepsilon))$. In particular, $l(X(\varepsilon), Y(\varepsilon))$ can be expressed in terms of $X(\varepsilon)$ and $Y(\varepsilon)$ with the help of only the operations of addition, multiplication by a scalar, and "$*$."

Proof. We use the notation of Paragraph 3 of Sec. 1. In this notation, $f(X(\varepsilon), Y(\varepsilon)) = e^{-L_{V(Y(\varepsilon))}}X(\varepsilon) + Y(\varepsilon)$. It is required to prove that $l(X(\varepsilon), Y(\varepsilon)) = e^{-L_{V(Y(\varepsilon))}}X(\varepsilon) + Y(\varepsilon)$. It is clear that it is sufficient to verify the last equation in the case when $X(\varepsilon) = \varepsilon\int_0^t X_\tau d\tau$, $Y(\varepsilon) = \varepsilon\int_0^t Y_\tau d\tau$, where $\int_0^t X_\tau d\tau$ and $\int_0^t Y_\tau d\tau$ are arbitrary nonstationary fields from $BD(\Phi)_T$.

The series $l\left(\varepsilon\int_0^t X_\tau d\tau,\ \varepsilon\int_0^t Y_\tau d\tau\right)$ is the MacLaurin decomposition with respect to ε of the family of fields

$$\int_0^t \left(\left(\overleftarrow{\exp}\int_0^\tau -\mathrm{ad}\,\varepsilon Y_\theta d\theta\right)\varepsilon X_\tau + \varepsilon Y_\tau\right) d\tau.$$

Hence, everything will be proved if we establish that the MacLaurin decomposition with respect to ε of the family $\int_0^t \left(\overleftarrow{\exp}\int_0^\tau -\mathrm{ad}\,\varepsilon Y_\theta d\theta\right)\varepsilon X_\tau d\tau$ coincides with the series $e^{-L_{V\left(\varepsilon\int_0^t Y_\tau d\tau\right)}}\int_0^t \varepsilon X_\tau d\tau.$

Let $Z(\varepsilon)_t$ be an arbitrary series from $BD(\Phi)_T[[\varepsilon]]$, so

$$L_{Z(\varepsilon)}\int_0^t \varepsilon X_\tau d\tau = Z(\varepsilon)_{t*}\int_0^t \varepsilon X_\tau d\tau = \varepsilon\int_0^t [Z(\varepsilon)_\tau, X_\tau]\, d\tau = \varepsilon\int_0^t (\mathrm{ad}\, Z(\varepsilon)_\tau) X_\tau d\tau.$$

Consequently,

$$e^{L_{Z(\varepsilon)_t}}\int_0^t \varepsilon X_\tau d\tau = \sum_{i=0}^{\infty}\frac{1}{i!}(L_{Z(\varepsilon)})^i\int_0^t \varepsilon X_\tau d\tau = \varepsilon\int_0^t e^{\mathrm{ad} Z(\varepsilon)_\tau} X_\tau d\tau.$$

In the last equation, setting $Z(\varepsilon)_t = -V\left(\varepsilon\int_0^t Y_\tau d\tau\right)$, we arrive at the conclusion that it is sufficient to prove the coincidence of the series $e^{-\mathrm{ad} V\left(\varepsilon\int_0^t Y_\tau d\tau\right)}$ with the MacLaurin decomposition in ε of the family $\left(\overleftarrow{\exp}\int_0^t -\mathrm{ad}\,\varepsilon Y_\tau d\tau\right) X_t$.

Finally, by virtue of the arbitrariness of the family X_t, one can assume that X_t is equal to the stationary (independent of t) field X_0. We denote the MacLaurin decomposition in ε of the family $\left(\overleftarrow{\exp}\int_0^t -\mathrm{ad}\,\varepsilon Y_\tau d\tau\right) X_0$ by $\mathscr{E}_t(\varepsilon)$. It is required to establish the equation in formal series

$$\mathscr{E}_t(\varepsilon) = e^{-\mathrm{ad} V\left(\varepsilon\int_0^t Y_\tau d\tau\right)} X_0.$$

Everywhere below, differentiation and integration of formal series are carried out termwise.

From the corresponding equation for the family $\overleftarrow{\exp}\int_0^t -\operatorname{ad} Y_\tau d\tau X_0$, we get that the series $\mathscr{E}_t(\varepsilon)$ satisfies the linear differential equation

$$\frac{d}{dt}\mathscr{E}_t(\varepsilon) = -\varepsilon(\operatorname{ad} Y_t)\mathscr{E}_t(\varepsilon), \quad \mathscr{E}_0(\varepsilon) = X_0. \tag{2.16}$$

It is easy to see that the series $\mathscr{E}_t(\varepsilon)$ is the unique formal series in ε satisfying this equation with initial condition X_0. In fact, the relation

$$\mathscr{E}_t(\varepsilon) = X_0 - \varepsilon\int_0^t (\operatorname{ad} Y_\tau)\mathscr{E}_\tau(\varepsilon)\,d\tau,$$

equivalent with the differential equation, uniquely determines the coefficient of the power ε^{m+1} in the series $\mathscr{E}_t(\varepsilon)$ in terms of the coefficient of ε^m, $\forall m \geqslant 0$.

Thus, Proposition 2.3 will be proved if we establish that the series $e^{-\operatorname{ad} V\left(\varepsilon\int_0^t Y_\tau d\tau\right)}X_0$ satisfies the linear differential equation (2.16). The initial condition is satisfied since $V(0) = 0$.

It remains to differentiate with respect to t the expression $e^{-\operatorname{ad} V\left(\varepsilon\int_0^t Y_\tau d\tau\right)}X_0$. We use the formula

$$\frac{d}{dt}e^{\operatorname{ad} Z_t(\varepsilon)}X_0 = \left(\operatorname{ad}\int_0^1 e^{\rho \operatorname{ad} Z_t(\varepsilon)}\frac{d}{dt}Z_t(\varepsilon)\,d\rho\right)\circ e^{\operatorname{ad} Z_t(\varepsilon)}X_0, \tag{2.17}$$

true for any series $Z_t(\varepsilon) \in BD(\Phi)_T[[\varepsilon]]$. This formula, in essence, is well known. Here is its derivation:

We fix t_1, $t_2 \in \mathbf{R}$ and we write $P_\rho(\varepsilon) = e^{\rho \operatorname{ad} Z_{t_2}(\varepsilon)}e^{-\rho \operatorname{ad} Z_{t_1}(\varepsilon)}$, where $\rho \in \mathbf{R}$. We have

$$\frac{d}{d\rho}P_\rho(\varepsilon) = e^{\rho \operatorname{ad} Z_{t_2}(\varepsilon)}\circ\operatorname{ad}(Z_{t_2}(\varepsilon) - Z_{t_1}(\varepsilon))\circ e^{-\rho \operatorname{ad} Z_{t_1}(\varepsilon)} =$$

$$= P_\rho(\varepsilon)\circ e^{\rho \operatorname{ad} Z_{t_1}(\varepsilon)}\circ\operatorname{ad}(Z_{t_2}(\varepsilon) - Z_{t_1}(\varepsilon))\circ e^{-\rho \operatorname{ad} Z_{t_1}(\varepsilon)} = P_\rho(\varepsilon)\circ\left(\left(\operatorname{Ad} e^{\rho \operatorname{ad} Z_{t_1}(\varepsilon)}\right)\operatorname{ad}(Z_{t_2}(\varepsilon) - Z_{t_1}(\varepsilon))\right)$$

$$= P_\rho(\varepsilon)\circ\left(e^{\operatorname{ad}(\rho \operatorname{ad} Z_{t_1}(\varepsilon))}\operatorname{ad}(Z_{t_2}(\varepsilon) - Z_{t_1}(\varepsilon))\right) = P_\rho(\varepsilon)\circ\left(\operatorname{ad}\left(e^{\rho \operatorname{ad} Z_{t_1}(\varepsilon)}(Z_{t_2}(\varepsilon) - Z_{t_1}(\varepsilon))\right)\right).$$

The last equation follows from the identity

$$[\operatorname{ad} Z^1, \operatorname{ad} Z^2] = \operatorname{ad}[Z^1, Z^2],$$

if one notes that $(\operatorname{ad}(\operatorname{ad} Z^1))\operatorname{ad} Z^2 \overset{\text{def}}{=} [\operatorname{ad} Z^1, \operatorname{ad} Z^2]$.

The differential equation for $P_\rho(\varepsilon)$ can be rewritten in the form

$$P_\rho(\varepsilon) = \operatorname{Id} + \int_0^\rho P_{\rho'}(\varepsilon)\circ\operatorname{ad}\left(e^{\rho' \operatorname{ad} Z_{t_1}(\varepsilon)}(Z_{t_2}(\varepsilon) - Z_{t_1}(\varepsilon))\right)d\rho'.$$

Setting $\rho = 1$ and recalling the definition of $P_\rho(\varepsilon)$, we get

$$e^{\operatorname{ad} Z_{t_2}(\varepsilon)} = e^{\operatorname{ad} Z_{t_1}(\varepsilon)} + \int_0^1 P_\rho(\varepsilon)\circ\operatorname{ad}\left(e^{\rho \operatorname{ad} Z_{t_1}(\varepsilon)}(Z_{t_2}(\varepsilon) - Z_{t_1}(\varepsilon))\right)d\rho\, e^{\operatorname{ad} Z_{t_1}(\varepsilon)}.$$

Letting t_2 tend to t_1, we get (2.17) instantly. If in (2.17) we integrate with respect to ρ, we get the equation

$$\frac{d}{dt}e^{\operatorname{ad} Z_t(\varepsilon)}X_0 = \left(\operatorname{ad}\left(\frac{e^{\operatorname{ad} Z_t(\varepsilon)} - \operatorname{Id}}{\operatorname{ad} Z_t(\varepsilon)}\frac{d}{dt}Z_t(\varepsilon)\right)\right)\circ e^{\operatorname{ad} Z_t(\varepsilon)}X_0,$$

where $\frac{e^{\operatorname{ad} Z_t(\varepsilon)} - \operatorname{Id}}{\operatorname{ad} Z_t(\varepsilon)}$ is symbolic notation for the series $\sum_{m=0}^{\infty}\frac{1}{(m+1)!}(\operatorname{ad} Z_t(\varepsilon))^m$. Using the series χ, introduced in Paragraph 3 of Sec. 1, this equation can be written again in the form

$$\frac{d}{dt}e^{\operatorname{ad} Z_t(\varepsilon)}X_0 = \left(\operatorname{ad}(1/\chi(\operatorname{ad} Z_t(\varepsilon))\frac{d}{dt}Z_t(\varepsilon))\right)\circ e^{\operatorname{ad} Z_t(\varepsilon)}X_0.$$

It still remains for us to calculate $\frac{d}{dt} V\left(\varepsilon \int_0^t Y_\tau d\tau\right)$.

In Sec. 1 [formula (1.5)] we got the equation for V,

$$V\left(\varepsilon \int_0^t Y_\tau d\tau\right) = \chi\left(L_{V\left(\varepsilon \int_0^t Y_\tau d\tau\right)}\right) \int_0^t \varepsilon Y_\tau d\tau = \int_0^t \chi\left(\operatorname{ad} V\left(\varepsilon \int_0^\tau Y_\theta d\theta\right)\right) \varepsilon Y_\tau d\tau.$$

Consequently,

$$\frac{d}{dt} V\left(\varepsilon \int_0^t Y_\tau d\tau\right) = \chi\left(\operatorname{ad} V\left(\varepsilon \int_0^t Y_\tau d\tau\right)\right) \varepsilon Y_t.$$

Finally, we get

$$\frac{d}{dt} e^{-\operatorname{ad} V\left(\varepsilon \int_0^t Y_\tau d\tau\right)} X_0 = -\left(\operatorname{ad}\left(1/\chi\left(\operatorname{ad} V\left(\varepsilon \int_0^t Y_\tau d\tau\right)\right)\right)\right.$$

$$\cdot \frac{d}{dt} V\left(\varepsilon \int_0^t Y_\tau d\tau\right)\right) \circ e^{-\operatorname{ad} V\left(\varepsilon \int_0^t Y_\tau d\tau\right)} X_0 = -\varepsilon \operatorname{ad} Y_t \circ e^{-\operatorname{ad} V\left(\varepsilon \int_0^t Y_\tau d\tau\right)} X_0.$$

Proposition 2.3 is proved.

Remark. The series $V\left(\varepsilon \int_0^t Y_\tau d\tau\right)$, involved in the definition of $f\left(\varepsilon \int_0^t X_\tau d\tau, \varepsilon \int_0^t Y_\tau d\tau\right)$, in general, diverges, even if the field Y_t is analytic. Nevertheless, in certain cases convergence holds all the same. Let, for example, $\mathcal{B}$ be some subalgebra of the Lie algebra $D(\Psi)$, where on $\mathcal{B}$ there is given a norm $\|\cdot\|$, $\|[X, Y]\| \leqslant \|X\| \cdot \|Y\|$ $\forall X, Y \in \mathcal{B}$, turning $\mathcal{B}$ into a Banach Lie algebra. If the nonstationary field Y_t is such that $Y_t \in \mathcal{B}$ $\forall t \in \mathbf{R}$, then the series $V\left(\varepsilon \int_0^t Y_\tau d\tau\right)$ converges absolutely in $\mathcal{B}$ for all sufficiently small ε. In [3] the convergence of this is proved for $\varepsilon \int_0^t \|Y_\tau\| d\tau \leqslant 0.44$. As S. Vakhrameev proved, this estimate can be improved. Namely, the series converges for

$$\varepsilon \int_0^t \|Y_\tau\| d\tau \leqslant \int_0^\pi \frac{2 d\theta}{\theta(1 - \operatorname{ctg}\theta) + 2}.$$

The last estimate is, apparently, sharp.

3. Nilpotence. Integrability. In this paragraph, there are described briefly some applications of the results obtained. The proofs are only outlined. A detailed account (together with other applications) will be given in later publications.

I. The considerations given above allow one to answer the question: in which case of integrating the nonautonomous differential equation on M

$$\frac{d}{dt} \hat{P}_t = \hat{P}_t \circ X_t, \quad \hat{P}_0 = \mathrm{Id}, \quad \int_0^t X_\tau d\tau \in D(\Phi)_T \tag{2.18}$$

can one reduce to integrating autonomous differential equations on the same manifold M. The precise result will be formulated somewhat later, but first we introduce the following notation.

Let $D(\Phi) \ni Y$ be a vector field (stationary), $\|Y\|_s < \infty$, $s \geq 0$. We write

$$e^{tY} = \overrightarrow{\exp} \int_0^t Y d\tau.$$

Thus, e^{tY} is a solution of the autonomous differential equation

$$\frac{\partial}{\partial t} \hat{Q}_t = \hat{Q}_t \circ Y, \quad \hat{Q}_0 = \mathrm{Id}.$$

Analogously we define

$$e^{t\,\mathrm{ad}\,Y} = \overrightarrow{\exp}\int_0^t \mathrm{ad}\,Y\,d\tau.$$

It is easy to see that

$$e^{t_1 Y}\circ e^{t_2 Y} = e^{(t_1+t_2)Y}, \quad \forall t_1,\ t_2 \in \mathbf{R}.$$

In particular, $e^{-Y} = (e^{Y})^{-1}$. Consequently,

$$\overrightarrow{\exp}\int_0^t Y\,d\tau = e^{tY} = \overleftarrow{\exp}\int_0^t Y\,d\tau.$$

Analogous identities are also valid for $e^{t\,\mathrm{ad}\,Y}$.

Let Y_t be a family of vector fields from $D(\Phi)$ which is differentiable with respect to $t \in \mathbf{R}$. There holds

$$\frac{d}{dt}e^{Y_t} = e^{Y_t}\circ\int_0^1 e^{-\tau\,\mathrm{ad}\,Y_t}\frac{d}{dt}Y_t\,d\tau. \tag{2.19}$$

This equation can be derived without difficulty from (2.9) for the chronological logarithm of the composition of two flows. In fact, let $t \in \mathbf{R}$ be fixed and $\delta_\varepsilon Y = Y_{t+\varepsilon} - Y_t$. We have

$$e^{Y_t+\delta_\varepsilon Y} = \overleftarrow{\exp}\int_0^1 (Y_t+\delta_\varepsilon Y)\,d\tau = \overleftarrow{\exp}\int_0^1 Y_t\,d\tau\circ\overleftarrow{\exp}\int_0^1\left(\overrightarrow{\exp}\int_0^\tau -\mathrm{ad}\,Y_t\,d\theta\,\delta_\varepsilon Y\right)d\tau = e^{Y_t}\circ\overleftarrow{\exp}\int_0^1 e^{-\tau\,\mathrm{ad}\,Y_t}\delta_\varepsilon Y\,d\tau.$$

Differentiating the last equation with respect to ε for $\varepsilon = 0$, we get (2.19).

Let us assume now that the family Y_t is such that

$$\int_0^1 e^{-\tau\,\mathrm{ad}\,Y_t}\frac{d}{dt}Y_t\,d\tau = X_t \quad \forall t \in \mathbf{R}, \quad Y_0 = 0. \tag{2.20}$$

Then the flow e^{Y_t}, $t \in \mathbf{R}$, satisfies the equation

$$\frac{d}{dt}e^{Y_t} = e^{Y_t}\circ X_t, \quad e^{Y_0} = \mathrm{Id}.$$

Consequently,

$$e^{Y_t} = \overrightarrow{\exp}\int_0^t X_\tau\,d\tau.$$

We shall give conditions sufficient for (2.20) to have a solution Y_t which can be expressed in terms of X_t and quadratures. As a preliminary, we recall some algebraic concepts.

An arbitrary algebra $\mathfrak{A}$ is called nilpotent if there exists an integer $n > 0$ such that the product of any $\geq n$ elements of the algebra $\mathfrak{A}$ (independent of the arrangement of parentheses) is equal to zero. The least integer n for which the indicated property holds is called the length of nilpotency of the algebra $\mathfrak{A}$.

By a nonassociative polynomial in one variable with real coefficients is meant an arbitrary element of the free algebra over $\mathbf{R}$ with one generator.

Let p be some nonassociative polynomial and x be an element of the given algebra $\mathfrak{A}$. By the value $p(x)$ of the polynomial p at the element x is meant the image of p under the homomorphism of the free algebra into $\mathfrak{A}$, carrying the generator of the free algebra into x. It is clear that if the algebra $\mathfrak{A}$ is generated by the element x, then any element of $\mathfrak{A}$ has the form $p(x)$, where p is some nonassociative polynomial.

Let us assume now that the ch-subalgebra of $D(\Phi)_T$ generated by the nonstationary field $\tilde{X} = \int_0^t X_\tau\,d\tau$ is nilpotent. We denote this algebra by $\mathfrak{A}_X$ and we shall solve (2.20) with respect to $Y_t \in \mathfrak{A}_X$.

Equation (2.20) is equivalent in the nilpotent algebra $\mathfrak{A}_X$ with the equation

$$\frac{1-e^{-L_Y}}{L_Y}Y = \tilde{X},$$

where $\frac{1-e^{-L_Y}}{L_Y}=\frac{1}{\chi}(L_Y)=\sum_{m=0}^{\infty}\frac{1}{(m+1)!}(-L_Y)^m$.

In this sum only a finite number of terms are different from zero, since the algebra $\mathfrak{A}_X$ is nilpotent. Our equation can be rewritten in the form

$$Y=\chi(L_Y)\tilde{X}.$$

We have already met such an equation [cf. (1.5)]. It has a solution in $\mathfrak{A}_X$. If the nilpotency length of $\mathfrak{A}_X$ is equal to n, then a solution is (see p. 1657).

$$Y=\sum_{i=1}^{n-1}V_i(\tilde{X}).$$

We summarize what has been said:

Proposition 2.4. Let us assume that the ch-subalgebra $\mathfrak{A}_X$ of $D(\Phi)_T$, generated by the field $\int_0^t X_\tau d\tau$, is nilpotent with nilpotency length n. Then

$$\overrightarrow{\exp}\int_0^t X_\tau d\tau=e^{\sum_{i=1}^{n}V_i\left(\int_0^t X_\tau d\tau\right)},\quad t\in\mathbf{R}.$$

The homogeneous nonassociative polynomials V_i can be calculated recursively, starting from (1.5). The results of such a calculation for degrees 1-4 are given on p.1657.

There is an effective criterion for verifying whether a given ch-algebra with one generator is nilpotent of length n. In Paragraph 2 of Sec. 1, there is described a method of constructing a basis for a free ch-algebra with one generator. Let p_{mi} $(m = 1, 2, \ldots; i = 1, \ldots, b_1^{(m)})$ be nonassociative polynomials defining such a basis. From the results of Paragraph 2 of Sec. 1 one can deduce the following criterion.

For a ch-algebra with one generator x to be nilpotent with nilpotency length $\leq n$, it is necessary and sufficient that one have

$$p_{mi}(x)=0 \text{ for } n\leqslant m\leqslant \max\{2n-3;\, n\},\; i=1,\ldots,b_1^{(m)}.$$

For n = 2 and n = 3, we get the corollary to Proposition 2.4:

a) (n = 2) if $\left[\int_0^t X_\tau d\tau, X_t\right]=0\;\forall t\in\mathbf{R}$, then

$$\overrightarrow{\exp}\int_0^t X_\tau d\tau=e^{\int_0^t X_\tau d\tau};$$

b) (n = 3) if $\left[\int_0^t X_\tau d\tau, \left[\int_0^t X_\tau d\tau, X_t\right]\right]=0$ and $\left[\int_0^t\left[\int_0^\tau X_\theta d\theta, X_\tau\right]d\tau, X_t\right]=0\;\forall t\in\mathbf{R}$, then

$$\overrightarrow{\exp}\int_0^t X_\tau d\tau=e^{\int_0^t X_\tau d\tau+\frac{1}{2}\int_0^t\left[\int_0^\tau X_\theta d\theta, X_\tau\right]d\tau}.$$

Remark. Let us assume that $M = \mathbf{R}^k$, and the function $X_tE(x)$ depends linearly on $x\in\mathbf{R}^k$, $X_tE(x) = A_tx$, where A_t is a family of k × k-matrices. Then the flow $P_t(x)$, which is a solution of (2.18), is a family of linear transformations of the space $\mathbf{R}^k$, $P_t(x) = U_tx$, where U_t is a family of nondegenerate k × k-matrices. Here (2.18) reduces to the linear matrix equation

$$\frac{dU}{dt}=A_tU. \tag{2.21}$$

In this case Proposition 2.4 gives a sufficient condition for integrability of the nonautonomous equation (2.21) by quadratures, since linear autonomous equations are integrable by quadratures.

II. One can generalize the concept of distribution of planes to manifolds, introducing nonstationary distributions. A certain algebraic interpretation of the concept of complete integrability of a distribution of planes allows one to introduce the corresponding concept also for nonstationary distributions. The results of Paragraph 2 of the present section lead to conditions for the complete integrability of nonstationary distributions analogous to the Frobenius conditions in the stationary case.

Suppose given a distribution of planes $\mathscr{P}=\{\Pi(x)\mid x\in M\}$ on the manifold M, $\Pi(x)\subset T_xM$ $\forall x\in M$. We associate with the distribution $\mathscr{P}$ the space of nonstationary fields $\mathscr{X}_T$ consisting of those fields $\int_0^t X_\tau d\tau\in BD(\Phi)_T$ such that $X_tE(x)\in\Pi(x)$ $\forall x\in M$, $\forall t\in \mathbf{R}$.

We recall that a distribution of planes $\mathscr{P}$ is called completely integrable if through each point $x \in M$ there passes a submanifold $N_x\subset M$ such that for any $y \in N_x$, the plane Π_y is the tangent plane to N_x at the point y. With the help of the space $\mathscr{X}_T$, this definition can be reformulated in the following way.

The distribution of planes $\mathscr{P}$ is called completely integrable if the set of flows

$$\left\{\overrightarrow{\exp}\int_0^t X_\tau d\tau \,\middle|\, \int_0^t X_\tau d\tau\in\mathscr{X}_T\right\}$$

is a subgroup of the group of all flows. The fact that from complete integrability in the sense of the first definition follows complete integrability in the sense of the second is obvious. To establish the reverse implication is also easy: let $x \in M$, $\xi_1, \ldots, \xi_k$ be a basis for the vector space $\Pi(x)$ and $X_1, \ldots, X_k$ be vector fields on M such that $X_iE(y)\in\Pi(y)$ $\forall y\in M$ and $X_iE(x) = \xi_i$, $i = 1, \ldots, k$. The map

$$(s_1,\ldots,s_k)\to e^{s_1X_1}\circ\ldots\circ e^{s_kX_k}E(x)$$

defines local coordinates on a certain submanifold of M. If the distribution $\mathscr{P}$ is completely integrable in the sense of the second definition, then, as is easy to see, the vectors $(\partial/\partial s_i)e^{s_1X_1}\circ\ldots\circ e^{s_kX_k}E(x)$ lie in $\Pi(e^{s_1X_1}\circ\ldots\circ e^{s_kX_k}E(x))$ for $i = 1, \ldots, k$. Consequently, our submanifold is the one sought and the distribution $\mathscr{P}$ is completely integrable in the sense of the first definition.

Definition. By a nonstationary distribution on a manifold M is meant an arbitrary vector subspace $\mathscr{X}\subset BD(\Phi)_T$, closed in the topology defined by the family of seminorms $\|\cdot\|_{s,K}^{t_1,t_2}$, where

$$\left\|\int_0^t X_\tau d\tau\right\|_{s,K}^{t_1,t_2}=\int_{t_1}^{t_2}\|X_\tau\|_{s,K}d\tau,\quad t_1,t_2\in\mathbf{R},\quad s\geqslant 0,$$

K is a compactum in M.

A nonstationary distribution $\mathscr{X}$ is called completely integrable if the set of flows

$$\left\{\overrightarrow{\exp}\int_0^t X_\tau d\tau \,\middle|\, \int_0^t X_\tau d\tau\in\mathscr{X}\right\}$$

forms a subgroup of the group of all flows.

Proposition 2.5. If the nonstationary distribution $\mathscr{X}$ is completely integrable, then the space $\mathscr{X}$ is a subalgebra of the ch-algebra $BD(\Phi)_T$.

Proof. Let $\int_0^t X_\tau d\tau$, $\int_0^t Y_\tau d\tau\in\mathscr{X}$. From the definition of complete integrability it follows that the field

$$\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t \varepsilon X_\tau d\tau\circ\overrightarrow{\exp}\int_0^t \varepsilon Y_\tau d\tau\right)$$

also belongs to $\mathscr{X}$ for any $\varepsilon \in \mathbf{R}$. On the other hand,

$$\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t \varepsilon X_\tau d\tau\circ\overrightarrow{\exp}\int_0^t \varepsilon Y_\tau d\tau\right)=\varepsilon\left(\int_0^t X_\tau d\tau+\int_0^t Y_\tau d\tau\right)-\varepsilon^2\left(\int_0^t Y_\tau d\tau\right)*\left(\int_0^t X_\tau d\tau\right)+O(\varepsilon^3)\quad(\varepsilon\to 0).$$

Consequently, the field $\left(\int_0^t Y_\tau d\tau\right)*\left(\int_0^t X_\tau d\tau\right)$ belongs to $\mathscr{X}$.

Remark. The proposition proved gives a necessary condition for the complete integrability of nonstationary distributions. Is this condition sufficient? Let $\mathscr{X}$ be some nonstationary distribution, where the space

$\mathscr{X}$ is a subalgebra of the ch-algebra $BD(\Phi)_T$. In Paragraph 2 of the present section (cf. Proposition 2.3) it was proved that for $\int_0^t X_\tau d\tau,\ \int_0^t Y_\tau d\tau \in \mathscr{X}$ the field $\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right)$, at least asymptotically, can be expressed in terms of $\int_0^t X_\tau d\tau$ and $\int_0^t Y_\tau d\tau$ with the help of the operations of the ch-algebra $\mathscr{X}$. Thus, if the corresponding asymptotic expansion converges to $\overrightarrow{\ln}\left(\overrightarrow{\exp}\int_0^t X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t Y_\tau d\tau\right)\ \forall \int_0^t X_\tau d\tau,\ \int_0^t Y_\tau d\tau \in \mathscr{X}$ then the nonstationary distribution $\mathscr{X}$ is completely integrable. However, this asymptotic expansion in general may diverge. Convergence holds in the case when the manifold M is analytic, the distribution $\mathscr{X}$ consists of analytic vector fields, and satisfies strong restrictions. The precise description of the corresponding spaces of analytic fields leaves the domain of the present paper.

LITERATURE CITED

1. A. A. Agrachev, "A necessary condition for optimality of the second order in the general nonlinear case," Mat. Sb., 102, No. 4, 551-568 (1977).
2. A. A. Agrachev and R. V. Gamkrelidze, "A second order optimality principle for high speed problems," Mat. Sb., 100, No. 4, 610-643 (1976).
3. A. A. Agrachev and R. V. Gamkrelidze, "Exponential representation of flows and the chronological calculus," Mat. Sb., 107, No. 4, 467-532 (1978).
4. R. Gabasov and F. M. Kirillova, Singular Optimal Equations [in Russian], Nauka, Moscow (1973).
5. J.-P. Serre, Lie Algebras and Lie Groups, W. A. Benjamin (1965).
6. R. Gamkrelidze, "Exponential representation of solutions of ordinary differential equations," Proc. Equadiff. Fifth Conference Prague 1977, Springer-Verlag (1978).
7. H. Hermes, "Local controllability and sufficient conditions in singular problems. I; II," J. Diff. Eq., 20, No. 1, 213-232 (1976); SIAM J. Control., 14, No. 6, 1049-1062 (1976).
8. H. J. Kelly, R. E. Kopp, and H. G. Moyer, Singulary Extremals. Topics in Optimization, Academic Press (1967).
9. H. W. Knobloch, "High order necessary conditions and local controllability," Math. Inst. der Univ. Würzburg (1978) (preprint).
10. A. J. Krener, "The high order maximal principle and its applications to singular extremals," SIAM J. Control, 15, No. 2, 256-293 (1977).