The Cheeger problem in perimeter-measure spaces and applications

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References

- V. Franceschi, A. Pinamonti, G. Saracco and G. Stefani, The Cheeger problem in abstract measure spaces JLMS 109 (2024), no. 1, arXiv:2207.00482

- Project on Distributional fractional BV and Sobolev spaces [9 papers] in collaboration with Comi (and Alicandro, Braides, Bruè, Calzi, Solci, Spector)

- K. Bessas and G. Stefani, Non-local BV functions and a denoising model with L^1 fidelity, ACV (2024), ahead of print, arXiv:2210.11958

- G. Saracco and G. Stefani, On the N-Cheeger problem for component-wise increasing norms, submitted paper, arXiv:2401.16041

Warm up: Cheeger problem in $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be a non-empty bounded open set with Lipschitz boundary.

The Cheeger problem is the isoperimetric-type optimization problem

$$h(\Omega) = \inf\left\{\frac{P(E)}{|E|} : E \subset \Omega,\ |E| > 0\right\} \in [0, \infty).$$

The number $h(\Omega)$ is the Cheeger constant and any minimizer is a Cheeger set of Ω.

This problem was introduced by Maz'ya (1962) and Cheeger (1969) and links to:

- lower bounds for the first eigenvalue of the Dirichlet p-Laplacian operator
- the creep torsion problem
- the existence of sets with prescribed constant mean curvature
- total variation denoising models
- the minimum flow-maximum cut problem
- elasto-plastic models of plate failure
- Bingham fluids and landslide models
- an elementary proof of the Prime Number Theorem (Ω = a square)

Generalization: cluster Cheeger sets after [Caroccia-Littig] and [Bobkov-Parini].

An abstract formulation

The definition of the Cheeger problem only requires two ingredients:

a measure space $(X, \mathscr{A}, \mathfrak{m})$ and a perimeter functional $P\colon \mathscr{A} \to [0, \infty]$.

Indeed, one can consider a set $\Omega \in \mathscr{A}$ and define

$$h(\Omega) = \inf\left\{ \frac{P(E)}{\mathfrak{m}(E)} : E \subset \Omega,\ \mathfrak{m}(E) > 0,\ P(E) < \infty \right\}.$$

Well-posedness: P is proper (i.e., $P \not\equiv \infty$) and Ω is admissible (the inf set $\neq \emptyset$).

Main idea: treat local (weighted, Riemannian, sub-Riemannian, **CD**, discrete) and non-local (fractional, distributional) perimeter functionals at the same time.

Question: which assumptions on $(X, \mathscr{A}, \mathfrak{m})$ and P do ensure:

- existence of Cheeger sets?
- the relation with the first 1-eigenvalue?
- the relation with the first p-eigenvalue for $p \in (1, \infty)$? [local perimeters]

Related literature on abstract formulations

An abstract point of view has already be considered in the literature:

- Buttazzo & Velichkov, Shape optimization problems on metric measure spaces, J. Funct. Anal. (2013)
- Chambolle, Morini, Ponsiglione, Nonlocal curvature flows, ARMA (2015)
- Barozzi & Massari, Variational mean curvatures in abstract measure spaces, Calc. Var. PDE (2016)
- Górny & Mazón, The Anzellotti-Gauss-Green formula and least gradient functions in metric measure spaces, ESAIM COCV (2021)
- Buffa, Kinnunen & Pacchiano Camacho, Variational solutions to the total variation flow on metric measure spaces Nonlinear Anal. (2021)
- Buffa, Collins & Pacchiano Camacho, Existence of parabolic minimizers to the total variation flows on metric measure spaces, Manuscripta Math. (2022)
- Novaga, Paolini, Stepanov & Tortorelli, Isoperimetric clusters in homogeneous spaces via concentration compactness, J. Geom. Anal. (2022)

Existence of Cheeger sets

The existence of Cheeger sets follows from a simple compactness argument relying on lower semicontinuity, compactness and isoperimetric properties of P:

(lsc) P is lower semicontinuous w.r.t. $L^1(X, \mathfrak{m})$ convergence

(comp) $\{E \subset \Omega,\ \mathfrak{m}(\Omega) < \infty : P(E) \le c\}$ is compact in $L^1(X, \mathfrak{m})$

(isop) $\mathfrak{m}(E) \le \varepsilon \implies P(E) \ge f(\varepsilon)\,\mathfrak{m}(E)$ with $\lim_{\varepsilon \to 0^+} f(\varepsilon) = \infty$

Existence of Cheeger sets

Let $\Omega \in \mathscr{A}$ be an admissible set with $\mathfrak{m}(\Omega) \in (0, \infty)$.

(lsc) + (comp) + (isop) $\implies$ Ω admits Cheeger sets

Proof. Pick a minimizing sequence $(E_k)_k$. Since $P(E_k) \le 2\mathfrak{m}(E_k)\,h(\Omega)$, by (comp) we find a limit set $E \subset \Omega$. By (isop) we have $\mathfrak{m}(E) \in (0, \mathfrak{m}(\Omega)]$. By (lsc) we get that E is a minimizer. QED

Basic properties

Basic properties of the Cheeger constant and of Cheeger sets exploit the above assumptions and, possibly, the sub-modularity property of P:

$$\text{(sub)} \quad P(E \cap F) + P(E \cup F) \leq P(E) + P(F)$$

Basic properties of Cheeger constant

- $\Omega_1 \subset \Omega_2 \implies h(\Omega_1) \geq h(\Omega_2)$
- (isop) : $\mathfrak{m}(\Omega_k) \to 0^+ \implies h(\Omega_k) \to \infty$
- (lsc) + (comp) + (isop): $\Omega_k \xrightarrow{L^1} \Omega$, $\mathfrak{m}(\Omega) \in (0, \infty) \implies h(\Omega) \leq \liminf_k h(\Omega_k)$
  + (sub): $P(\Omega_k) \to P(\Omega) < \infty \implies h(\Omega) = \lim_k h(\Omega_k)$

Basic properties of Cheeger sets

- (isop): $\mathfrak{m}(E) \geq c$ for all Cheeger sets E, with $c = c(h(\Omega), f)$
- (sub): Cheeger sets are stable w.r.t. union and non-negligible intersection
  + (lsc): countable unions and non-negligible countable intersections
- (lsc) + (comp) + (isop) + (sub): minimal and maximal Cheeger sets

BV functions via coarea formula

The (total) variation of a measurable function $u \in L^0(X, \mathfrak{m})$ is

$$\mathrm{Var}(u) = \begin{cases} \int_{\mathbb{R}} P(\{u > t\})\, dt & \text{if } t \mapsto P(\{u > t\}) \text{ is } \mathscr{L}^1\text{-measurable} \\ \infty & \text{otherwise} \end{cases}$$

following the idea of [Visintin] and [Chambolle-Giacomini-Lussardi], so that

$$BV(X, \mathfrak{m}) = \{u \in L^1(X, \mathfrak{m}) : \mathrm{Var}(u) < \infty\}.$$

(empty) $P(\emptyset) = 0$ (space) $P(X) = 0$

Properties of variation

- $\mathrm{Var}(\lambda u) = \lambda \mathrm{Var}(u)$ for $\lambda > 0$ and $\mathrm{Var}(u + c) = \mathrm{Var}(u)$
- (empty) + (space): $\mathrm{Var}(c) = 0$ and $\mathrm{Var}(\chi_E) = P(E)$
- (lsc): Var is lower semicontinuous w.r.t. $L^1(X, \mathfrak{m})$ convergence
- (empty) + (space) + (sub) + (lsc): Var is convex on $L^1(X, \mathfrak{m})$

In particular, last point implies $BV(X, \mathfrak{m})$ is a convex cone in $L^1(X, \mathfrak{m})$.

The first 1-eigenvalue

Define $BV_0(\Omega) = \{u \in BV(X, \mathfrak{m}) : u = 0$ $\mathfrak{m}$-a.e. in $X \setminus \Omega\}$ and do not care of $\partial\Omega$.

Assume (empty) + (space), Ω admissible (so $BV_0(\Omega, \mathfrak{m}) \neq \{0\}$), $\mathfrak{m}(\Omega) \in (0, \infty)$.

The first 1-eigenvalue of Ω is (we allow sign-changing functions!)

$$\lambda_{1,1}(\Omega) = \inf\left\{ \frac{\mathrm{Var}(u)}{\|u\|_1} : u \in BV_0(\Omega, \mathfrak{m}),\ \|u\|_1 > 0 \right\} \in [0, \infty).$$

(sym) $\quad P(X \setminus E) = P(E)$

Symmetric coarea formula

$$\text{(lsc) + (sym)} \implies \mathrm{Var}(u) = \int_{-\infty}^{0} P(\{u < t\})\, dt + \int_{0}^{\infty} P(\{u > t\})\, dt$$

Link with first 1-eigenvalue

- $\lambda_{1,1}(\Omega) \leq h(\Omega)$
- (lsc) + (sym): $\lambda_{1,1}(\Omega) = h(\Omega)$

In particular, non-negligible level sets of minimizers of $\lambda_{1,1}(\Omega)$ are Cheeger sets.

Relative functionals

To deal with first p-eigenvalue, namely, Sobolev functions, we need local functionals.
We need a topological space $(X, \mathscr{T})$, $\mathscr{A} = \mathscr{B}(X)$ Borel σ-algebra and $\mathfrak{m}$ Borel.

We reinforce the (total) perimeter functional to the relative perimeter functional:

$$\mathscr{B}(X) \ni E \mapsto P(E; A) \text{ for any given open set } A \in \mathscr{T}.$$

The interesting properties now become:

(empty)$_R$ $\quad P(\emptyset; A) = 0$

(space)$_R$ $\quad P(X; A) = 0$

(sub)$_R$ $\quad P(E \cap F; A) + P(E \cup F; A) \leq P(E; A) + P(F; A)$

(lsc)$_R$ $\quad P(\,\cdot\,; A)$ is lower semicontinuous w.r.t. $L^1(X, \mathfrak{m})$ convergence

The relative variation is defined as before as

$$\mathrm{Var}(u; A) = \begin{cases} \displaystyle\int_{\mathbb{R}} P(\{u > t\}; A)\, dt & \text{if } t \mapsto P(\{u > t\}; A) \text{ is } \mathscr{L}^1\text{-measurable} \\ \infty & \text{otherwise} \end{cases}$$

and consequently we recover $BV(X, \mathfrak{m}) = \{u \in L^1(X, \mathfrak{m}) : \mathrm{Var}(u; X) < \infty\}$.

Perimeter and variation measures

Relative functionals $P(\cdot\,; A)$ and $\mathrm{Var}(\cdot\,; A)$ inherit same properties of total ones.

Perimeter measure

A set $E \in \mathscr{B}(X)$ has finite perimeter measure if $P(E;\cdot)\colon \mathscr{B}(X) \to [0,\infty)$ is a finite outer regular Borel measure on X.

Variation measure

A function $u \in L^0(X,\mathfrak{m})$ has finite variation measure if:

- the set $\{u > t\}$ has finite perimeter measure for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$;
- $\mathrm{Var}(u;\cdot)\colon \mathscr{B}(X) \to [0,\infty)$ is a finite outer regular Borel measure on X.

Notation: $u \in L^0(X,\mathfrak{m})$ has finite variation measure $\implies$ $\mathrm{Var}(u;\cdot) = |Du|(\cdot)$,

$$\mathsf{BV}(X,\mathfrak{m}) = \{u \in L^1(X,\mathfrak{m}) : u \text{ has finite variation measure}\} \subset BV(X,\mathfrak{m}).$$

Under (empty)$_\mathsf{R}$ + (space)$_\mathsf{R}$ + (sub)$_\mathsf{R}$ + (lsc)$_\mathsf{R}$ we have

$$|D(\lambda u)| = \lambda |Du| \text{ for } \lambda > 0, \quad |D(u+c)| = |Du|, \quad |Dc| = 0,$$

but $\mathsf{BV}(X,\mathfrak{m})$ may NOT be closed w.r.t. sum!

Example: intrinsic BV functions between subgroups in Carnot groups as in [Franchi-Serapioni-Serra Cassano] and [Di Donato-Le Donne].

Generalized coarea formula and chain rule

Generalized coarea formula

If $u \in L^0(X, \mathfrak{m})$ has finite variation measure, then

$$\int_A \varphi \, d|Du| = \int_{\mathbb{R}} \int_A \varphi \, d|D\chi_{\{u>t\}}| \, dt$$

for all $\varphi \in L^0(X, \mathfrak{m})$, $\varphi \geq 0$, and $A \in \mathscr{B}(X)$.

(local)$_R$ $\quad E \in \mathscr{T} \implies P(E; A) = 0$ for all $A \in \mathscr{B}(X)$ with $P(E; A \cap \partial E) = 0$

Chain rule

Assume (empty)$_R$ + (space)$_R$ + (local)$_R$ and let $\varphi \in C^1(\mathbb{R})$ be strictly increasing. If $u \in C(X)$ has finite variation measure, then $\varphi(u)$ has finite variation measure, with

$$|D\varphi(u)| = \varphi'(u) \, |Du|.$$

<u>Proof</u>. $\mathrm{Var}(\varphi(u); A) = \int_{\mathbb{R}} P(\{\varphi(u) > t\}; A) \, dt = \int_{\mathbb{R}} P(\{u > \varphi^{-1}(t)\}; A) \, dt =$

$$\int_{\mathbb{R}} \varphi'(s) \int_A d|D\chi_{\{u>s\}}| \, ds \overset{(\bigstar)}{=} \int_{\mathbb{R}} \int_A \varphi'(u) \, d|D\chi_{\{u>s\}}| \, ds \overset{\text{(coarea)}}{=} \int_A \varphi'(u) \, d|Du|$$

$(\bigstar)$: $\partial\{u > s\} \subset \{u = s\} \implies |D\chi_{\{u>s\}}|(A \cap \{u \neq s\}) = 0$. QED

Sobolev $W^{1,1}$ functions

Recall that

$$\mathsf{BV}(X,\mathfrak{m}) = \{u \in L^1(X,\mathfrak{m}) : u \text{ has finite variation measure}\} \subset BV(X,\mathfrak{m}),$$

so we can define

$$\mathsf{W}^{1,1}(X,\mathfrak{m}) = \{u \in \mathsf{BV}(X,\mathfrak{m}) : |Du| \ll \mathfrak{m}\} \subset \mathsf{BV}(X,\mathfrak{m})$$

so that

$$u \in \mathsf{W}^{1,1}(X,\mathfrak{m}) \implies |Du|(A) = \int_A |\nabla u|\, d\mathfrak{m} \quad \text{with } |\nabla u| \in L^1(X,\mathfrak{m}) \text{ the 1-slope}$$

Under $(\text{empty})_R$ + $(\text{space})_R$ + $(\text{sub})_R$ + $(\text{lsc})_R$ we have

$$|\nabla(\lambda u)| = \lambda |\nabla u| \text{ for } \lambda > 0, \quad |\nabla(u+c)| = |\nabla u|, \quad |\nabla c| = 0,$$

but $\mathsf{W}^{1,1}(X,\mathfrak{m})$ may NOT be closed w.r.t. sum! (recall intrinsic functions in groups)

From now on we shall assume that

$$(\text{sum})_R \quad u, v \in \mathsf{BV}(X,\mathfrak{m}) \implies u + v \in \mathsf{BV}(X,\mathfrak{m})$$

so that both $\mathsf{BV}(X,\mathfrak{m})$ and $\mathsf{W}^{1,1}(X,\mathfrak{m})$ are convex cones in $L^1(X,\mathfrak{m})$.

Sobolev $\mathsf{W}^{1,p}$ functions for $p \in (1, \infty)$

p-relaxed 1-slope

We say $G \in L^p(X, \mathfrak{m})$ is p-relaxed 1-slope of $u \in L^p(X, \mathfrak{m})$ if

$$\exists \{u_k\}_k \subset \mathsf{W}^{1,1}(X, \mathfrak{m}) \cap L^p(X, \mathfrak{m}) \quad \text{and} \quad \exists\, g \in L^p(X, \mathfrak{m})$$

such that

- $u_k \to u$ in $L^p(X, \mathfrak{m})$
- $|\nabla u_k| \in L^p(X, \mathfrak{m})$ and $|\nabla u_k| \rightharpoonup g$ weakly in $L^p(X, \mathfrak{m})$
- $g \leq G$ $\mathfrak{m}$-a.e. in X

From now on, we assume (empty)$_{\mathsf{R}}$ + (space)$_{\mathsf{R}}$ + (sub)$_{\mathsf{R}}$ + (lsc)$_{\mathsf{R}}$ + (sum)$_{\mathsf{R}}$, so that

$$\mathsf{Slope}_p(u) = \{G \in L^p(X, \mathfrak{m}) : G \text{ is a } p\text{-relaxed 1-slope of } u\}$$

is a (possibly empty) convex closed subset of $L^p(X, \mathfrak{m})$ for any $u \in L^p(X, \mathfrak{m})$.

We can thus define

$$\mathsf{W}^{1,p}(X, \mathfrak{m}) = \left\{u \in L^p(X, \mathfrak{m}) : \mathsf{Slope}_p(u) \neq \emptyset\right\}$$

and weak p-slope of $u \in \mathsf{W}^{1,p}(X, \mathfrak{m})$ is $|\nabla u|_p \in \mathsf{Slope}_p(u)$ with minimal L^p norm.

From now on $\Omega \subset X$ is a non-empty open set.

Strong approximation

If $u \in \mathsf{W}^{1,p}(X, \mathfrak{m})$ then $\exists \{u_k\}_k \subset \mathsf{W}^{1,1}(X, \mathfrak{m}) \cap L^p(X, \mathfrak{m})$ such that

$$|\nabla u_k| \in L^p(X, \mathfrak{m}) \quad \text{and} \quad u_k \to u, \ \ |\nabla u_k| \to |\nabla u|_p \text{ strongly in } L^p(X, \mathfrak{m}).$$

The space $W_0^{1,p}(\Omega, \mathfrak{m})$

We say that $u \in W_0^{1,p}(\Omega, \mathfrak{m})$ if $\exists \{u_k\}_k \subset \mathsf{W}^{1,1}(X, \mathfrak{m}) \cap L^p(X, \mathfrak{m})$ such that

- $|\nabla u_k| \in L^p(X, \mathfrak{m})$
- $u_k \to u$ and $|\nabla u_k| \to |\nabla u|_p$ strongly in $L^p(X, \mathfrak{m})$
- $u_k \in C(X)$ with $\operatorname{supp} u_k \subset \overline{\Omega}$

We say Ω is p-regular if $\mathsf{W}_0^{1,p}(\Omega, \mathfrak{m}) \neq \{0\}$.

The first p-eigenvalue of a p-regular Ω is

$$\lambda_{1,p}(\Omega) = \inf\left\{ \frac{\||\nabla u|_p\|_p^p}{\|u\|_p^p} : u \in \mathsf{W}_0^{1,p}(\Omega, \mathfrak{m}), \ \|u\|_p > 0 \right\} \in [0, \infty).$$

Link with the first p-eigenvalue

From now on $\Omega \subset X$ is a non-empty open p-regular set.

Link with first p-eigenvalue

$$\lambda_{1,p}(\Omega) \geq \left(\frac{\lambda_{1,1}(\Omega)}{p}\right)^p \quad \text{and } \lambda_{1,1}(\Omega) = h(\Omega) \text{ if } \Omega \text{ is admissible and } P(\,\cdot\,; X) \text{ is (sym)}$$

Proof. [Cheeger], [Lefton-Wei], [Kawohl-Fridman] Let $u \in \mathrm{W}_0^{1,p}(\Omega, \mathfrak{m})$ with $\|u\|_p > 0$.

Approximation: $\exists \{u_k\}_k \subset \mathrm{W}^{1,1}(X, \mathfrak{m}) \cap L^p(X, \mathfrak{m})$ with $|\nabla u_k| \in L^p(X, \mathfrak{m})$ such that $u_k \to u$, $|\nabla u_k| \to |\nabla u|_p$ strongly in $L^p(X, \mathfrak{m})$, and $u_k \in C(X)$, $\operatorname{supp} u_k \subset \overline{\Omega}$.

Let $\varphi(r) = r|r|^{p-1}$, $r \in \mathbb{R}$: $\varphi \in C^1(\mathbb{R})$ strictly increasing, $\varphi'(r) = p|r|^{p-1}$, $r \in \mathbb{R}$.

Chain rule: $\varphi(u_k) \in \mathrm{W}^{1,1}(X, \mathfrak{m}) \cap C(X)$ with $|\nabla \varphi(u_k)| = p|u_k|^{p-1}|\nabla u_k|$.

Since $\operatorname{supp} \varphi(u_k) \subset \overline{\Omega}$, we get $\varphi(u_k) \in BV_0(\Omega, \mathfrak{m})$, with

$$\mathrm{Var}(\varphi(u_k); X) = \||\nabla \varphi(u_k)|\|_1 = p \int_X |u_k|^{p-1} |\nabla u_k| \, d\mathfrak{m} \leq p\|u_k\|_p^{p-1} \||\nabla u_k|\|_p.$$

Therefore

$$\lambda_{1,1}(\Omega) \leq \frac{\mathrm{Var}(\varphi(u_k); X)}{\|\varphi(u_k)\|_1} \leq \frac{p\|u_k\|_p^{p-1} \||\nabla u_k|\|_p}{\|u_k\|_p^p} = p\,\frac{\||\nabla u_k|\|_p}{\|u_k\|_p}$$

and the conclusion follows letting $k \to \infty$ and then taking the inf. QED

Application # 1: Distributional fractional variation (Comi-S.) [1/2]

Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set.

For $s \in (0,1)$ and $p \in [1,\infty]$, the distributional s-variation of $u \in L^p(\mathbb{R}^n)$ in Ω is

$$|D^s u|(\Omega) = \sup\left\{\int_{\mathbb{R}^n} u \operatorname{div}^s \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n), \ \|\varphi\|_\infty \le 1, \ \operatorname{supp}\varphi \Subset \Omega\right\},$$

where the s-fractional divergence of φ is given by

$$\operatorname{div}^s \varphi(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+s+1}} \, dy, \qquad x \in \mathbb{R}^n.$$

We let $P(E;A) = |D^s \chi_E|(A)$, $\operatorname{Var}(u;A) = |D^s u|(A)$ for any open $A \subset \mathbb{R}^n$ and

$$BV^s(\Omega) = \left\{u \in L^1(\mathbb{R}^n) : |D^s u|(\Omega) < \infty\right\}.$$

Distributional meaning: $u \in L^1(\mathbb{R}^n)$ is in $BV^s(\Omega) \iff \exists\, D^s u \in \mathscr{M}(\Omega;\mathbb{R}^n)$ s.t.

$$\int_{\mathbb{R}^n} u \operatorname{div}^s \varphi \, dx = -\int_\Omega \varphi \cdot dD^s u \qquad \text{for all } \varphi \in C_c^\infty(\Omega;\mathbb{R}^n).$$

Fractional comparison: $W^{s,1}(\mathbb{R}^n) \subsetneq BV^s(\mathbb{R}^n)$ and $|D^s \chi_E|(\Omega) \le c_{n,s} P_s(E;\Omega)$.

Application #1: Distributional fractional variation (Comi-S.) [2/2]

Properties [Comi-S.]

- $|D^s \cdot|$ is translation invariant and $(n-s)$-homogeneous
- $|D^s\chi_\emptyset| = |D^s\chi_{\mathbb{R}^n}| = 0$, $|D^s\chi_{\mathbb{R}^n\setminus E}| = |D^s\chi_E|$
- $|D^s u|(\mathbb{R}^n) \le \liminf_k |D^s u_k|(\mathbb{R}^n) < \infty$ for $u_k \to u$ in $L^1(\mathbb{R}^n)$
- $\{u_k\}_k \subset BV^s(\mathbb{R}^n)$ bounded $\implies \exists \{u_{k_h}\}_h$ L^1_{loc}-converging to $u \in BV^s(\mathbb{R}^n)$
- $BV^s(\mathbb{R}^n) \subset L^{\frac{n}{n-s}}(\mathbb{R}^n)$ with $\|u\|_{L^{\frac{n}{n-s}}} \le c_{n,s}|D^s u|(\mathbb{R}^n)$ for $n \ge 2$

Bad news [Comi-S.]

- locality fails: $\exists\, \chi_E \in BV(\mathbb{R}^n)$ such that $\operatorname{supp}|D^s\chi_E| \not\subset \partial E$ (but $\subset \mathscr{F}^s E$)
- coarea formula fails: $\exists\, u \in BV^s(\mathbb{R}^n)$ with $t \mapsto |D^s\chi_{\{u>t\}}|(\mathbb{R}^n) \notin L^1(\mathbb{R})$
- submodularity of perimeter is unknown

Applications

- Cheeger sets exist in any open set $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$
- CMC sets in Ω exist for $\kappa \ge h(\Omega)$, being Cheeger sets for $\kappa = h(\Omega)$
- $\lambda_{1,1}(\Omega) \le h(\Omega)$, but the inequality may be strict

Application #2: Non-local variation (Bessas-S.) [1/3]

Let $K\colon \mathbb{R}^n \to [0,\infty]$ be a kernel.

The non-local K-variation of $u \in L^0(\mathbb{R}^n)$ is

$$[u]_K = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|\,K(x-y)\,dx\,dy$$

so that $BV^K(\mathbb{R}^n) = \{u \in L^1(\mathbb{R}^n) : [u]_K < \infty\}$. Notation: $P_K(E) = [\chi_E]_K$.

Examples

- $K \in L^1(\mathbb{R}^n) \implies BV^K(\mathbb{R}^n) = L^1(\mathbb{R}^n)$ [Mazón-Rossi-Toledo]
- $K = |\cdot|^{-n-s} \implies BV^K(\mathbb{R}^n) = W^{s,1}(\mathbb{R}^n)$ [Caffarelli-Roquejoffre-Savin]

We focus on the non-integrable case $K \notin L^1(\mathbb{R}^n)$ only.

Properties

- $[\,\cdot\,]_K$ is translation invariant, $P_K(\emptyset) = P_K(\mathbb{R}^n) = 0$, $P_K(\mathbb{R}^n \setminus E) = P_K(E)$
- $u_k \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n) \implies [u]_K \le \liminf_k [u_k]_K$ (Fatou)
- $[u \wedge v]_K + [u \vee v]_K \le [u]_K + [v]_K$

Application #2: Non-local variation (Bessas-S.) [2/3]

Isoperimetric inequality [Cesaroni-Novaga], [De Luca-Novaga-Ponsiglione]

K radially symmetric decreasing $\implies$ $P_K(E) \geq P_K(B^{|E|})$, with $|B^{|E|}| = |E|$

(with equality $\iff$ E is a ball, if K strictly decreasing in a ngbh of the origin)

Note that $\lim_{v \to 0^+} P_K(B^v)/v = \infty$ [Cesaroni-Novaga].

Compactness [Bessas-S.], [Foghem Gounoue]

$$K \notin L^1(\mathbb{R}^n), \quad K \in L^1(\mathbb{R}^n \setminus B_r) \text{ for all } r > 0$$

$$\Downarrow$$

$$\{u_k\}_k \subset BV^K(\mathbb{R}^n) \text{ bounded} \implies \exists \{u_{k_h}\}_h \ L^1_{loc}\text{-converging to } u \in BV^K(\mathbb{R}^n)$$

Application

- Cheeger sets exist in any open set $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$
- CMC sets in Ω exist for $\kappa \geq h(\Omega)$, being Cheeger sets for $\kappa = h(\Omega)$
- $\lambda_{1,1}(\Omega) = h(\Omega)$, with characterization of minimizers

Application #2: Non-local variation (Bessas-S.) [3/3]

Assume q-decreasing property: $|x| \le |y| \implies K(x)|x|^q \ge K(y)|y|^q$ for $q \ge 0$.

Monotonicity [Bessas-S.]

$$0 < r \le R < \infty \implies \frac{P_K(rE)}{|rE|^{2-\frac{q}{n}}} \ge \frac{P_K(RE)}{|RE|^{2-\frac{q}{n}}}$$

In particular, $v \mapsto P_K(B^v)v^{\frac{q}{n}-2}$ is decreasing (take $E = B$).

Isoperimetric inequality for small volumes [Bessas-S.]

K radial and $q < n+1$: $\quad |E| \le |B| \implies \dfrac{P_K(E)}{|E|^{2-\frac{q}{n}}} \ge \dfrac{P_K(B)}{|B|^{2-\frac{q}{n}}}$

A priori estimates [Bessas-S.]

Assume K radial and $q \in (n, n+1)$.

- $E \subset \Omega$ Cheeger set $\implies |E|^{\frac{q}{n}-1} \ge \dfrac{P_K(B^{|\Omega|})}{|\Omega|^{2-\frac{q}{n}}\, h(\Omega)}$
- $u \in BV_0^K(\Omega)$ eigenfunction $\implies \|u\|_{L^\infty(\Omega)} \le \left(\dfrac{|\Omega|^{2-\frac{q}{n}}\, h(\Omega)}{P_K(B^{|\Omega|})} \right)^{\frac{n}{q-n}} \|u\|_{L^1(\Omega)}$

Application #3: Minimal partition problems (Saracco-S.) [1/3]

Given $N \in \mathbb{N}$, an N-set of Ω is an N-tuple $\mathcal{E} = (\mathcal{E}_1, \dots, \mathcal{E}_N)$ of pairwise disjoint subsets with positive measure $\mathcal{E}_i \subset \Omega$, called chambers of E. Moreover, $\mathcal{E}$ is an N-cluster of Ω if $\mathcal{E}$ is an N-set with $P(\mathcal{E}_i) < \infty$ for each $i = 1, \dots, N$.

Fix a reference function $\Phi \colon \mathbb{R}^N_+ \to [0, \infty)$ (e.g., any q-norm) and $p \in [1, \infty)$.

Minimal partition problems

$$H^{\Phi,N}(\Omega) = \inf\left\{ \Phi\left(\frac{P(\mathcal{E}_1)}{|\mathcal{E}_1|}, \dots, \frac{P(\mathcal{E}_N)}{|\mathcal{E}_N|} \right) : \mathcal{E} \text{ is an } N\text{-cluster of } \Omega \right\}. \qquad (H)$$

$$\mathscr{L}^{\Phi,N}_{1,p}(\Omega) = \inf\{ \Phi\left(\lambda_{1,p}(\mathcal{E}_1), \dots, \lambda_{1,p}(\mathcal{E}_N)\right) : \mathcal{E} \text{ is an } N\text{-set of } \Omega \}, \qquad (\mathscr{L}_p)$$

$$\Lambda^{\Phi,N}_{1,p}(\Omega) = \inf\{ \Phi\left(\|\nabla u_1\|_p^p, \dots, \|\nabla u_N\|_p^p \right) : u = (u_1, \dots, u_N) \}, \qquad (\Lambda_p)$$

where the latter infimum runs on N-tuples $u = (u_1, \dots, u_N)$ of $W^{1,p}_0(\Omega)$ (or $BV_0(\Omega)$, for $p = 1$) functions with pairwise disjoint supports and unitary p-norm.

<u>Generalization of</u>: $\Phi = 1$-norm of [Caroccia-Littig], $\Phi = \infty$-norm of [Bobkov-Parini].

Application #3: Minimal partition problems (Saracco-S.) [2/3]

Assumptions on the reference function Φ

- Φ is continuous
- Φ is coercive, i.e., there exists $\delta > 0$ such that $\Phi(v) \geq \delta \|v\|_1$ for all $v \in \mathbb{R}^N_+$
- Φ is strictly increasing, i.e., if $v, w \in \mathbb{R}^N_+$ with $v < w$, then $\Phi(v) < \Phi(w)$

Existence of minimizers and correspondence [Saracco-S.]

Problems (H), $(\mathscr{L}_p)$ and (Λ_p) admit minimizers and

$$H^{\Phi,N}(\Omega) = \mathscr{L}_{1,1}^{\Phi,N}(\Omega) = \Lambda_{1,1}^{\Phi,N}(\Omega) \quad \text{and} \quad \mathscr{L}_{1,p}^{\Phi,N}(\Omega) = \Lambda_{1,p}^{\Phi,N}(\Omega) \quad \text{for } p > 1.$$

Moreover, minimizers are in a suitable 1-to-1 correspondence ($p = 1$ vs. $p > 1$).

Regularity of minimizers [Saracco-S.]

If $\partial\Omega \in C^1$, then (1-adj.) minimizers of (H) are open, $C^{1,\frac{1}{2}-}$ regular, meet $\partial\Omega$ tangentially and can be approximated by smooth N-clusters from the inside.

If $\Phi \in C^1$, then minimizers of (Λ_p) are bounded.

Application #3: Minimal partition problems (Saracco-S.) [3/3]

We have stability of constants and minimizers with respect to both p and Φ.

Stability as $p \to 1^+$ [Saracco-S.]

If $\partial\Omega \in C^1$, then $H^{\Phi,N}(\Omega) = \lim_{p\to 1^+} \mathscr{L}_{1,p}^{\Phi,N}(\Omega)$.

Moreover, any L^1-convergent minimizers converge to minimizers (up to inclusions).

A family $(\Phi_k)_{k\in\mathbb{N}}$ is equicoercive if $\inf_k \Phi_k(v) \geq \delta \|v\|_1$ for all $v \in \mathbb{R}^N_+$.

Stability as $\Phi_k \to \Phi$ [Saracco-S.]

If $(\Phi_k)_{k\in\mathbb{N}}$ is equicoercive and $\Phi = \lim_k \Phi_k = \Gamma\text{-}\lim_k \Phi_k$ is continuous, then

$$H^{\Phi,N}(\Omega) = \lim_k H^{\Phi_k,N}(\Omega),$$

$$\mathscr{L}_{1,p}^{\Phi,N}(\Omega) = \lim_k \mathscr{L}_{1,p}^{\Phi_k,N}(\Omega),$$

$$\Lambda_{1,p}^{\Phi,N}(\Omega) = \lim_k \Lambda_{1,p}^{\Phi_k,N}(\Omega).$$

Moreover, minimizers converge to minimizers (up to subsequences).

Remark: The results on minimal partitions generalize to perimeter-measure spaces!

Thank you for your kind attention!

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