ON THE MONOTONICITY OF PERIMETER ON NESTED CONVEX SETS

works with K. Bessas, F. Giannetti and G. Saracco (2017 – 2023)

MONOTONICITY

$E \subset \mathbb{R}^n$, $n \geq 2$ is a convex body $= E$ compact, convex, $\mathrm{int}(E) \neq \emptyset$

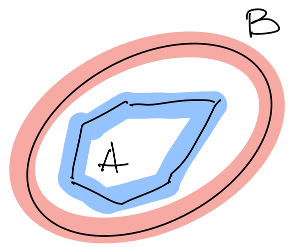
$E \subset \mathbb{R}^n$ convex body $\Longrightarrow P(E) = \mathcal{H}^{n-1}(\partial E)$

Hausdorff surface meas.

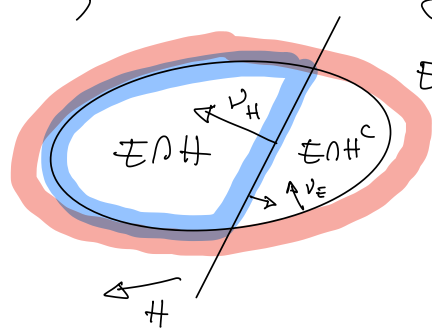
top. boundary

Theorem (Ancient Greek / Archimedes)

$A \subset B \subset \mathbb{R}^n$ convex bodies $\Longrightarrow P(A) \leq P(B)$



Proof. let $H =$ closed half-space, $E =$ convex body



Claim $P(E \cap H) \leq P(E)$

Write

inner normal of H

$$H = \{ x \in \mathbb{R}^n : x \cdot \nu_H \geq 0 \}$$

Set $X : \mathbb{R}^n \to \mathbb{R}^n$, $X \equiv -\nu_H$.

inn. norm. of $E \cap H^c$

$$0 = \int_{E \cap H^c} \mathrm{div}\, X \, dx = - \int_{\partial(E \cap H^c)} X \cdot \nu_{E \cap H^c} \, d\mathcal{H}^{n-1} \qquad \nu_{E \cap H^c} = \begin{cases} \nu_E & \text{on } \partial E \cap H^c \\ -\nu_H & \text{on } \partial H \cap E \end{cases}$$

$= \partial E \cap H^c \cup E \cap \partial H$

$$= -\int_{\partial E \cap H^c} X \cdot \nu_E \, d\mathcal{H}^{n-1} + \underbrace{\int_{\partial H \cap E} \underbrace{X \cdot \nu_H}_{=-1} \, d\mathcal{H}^{n-1}}_{-\mathcal{H}^{n-1}(\partial H \cap E)}$$

$$\wedge\!\mid$$

$$\mathcal{H}^{n-1}(\partial E \cap H^c)$$

$$\Longrightarrow \mathcal{H}^{n-1}(\partial H \cap E) \overset{(1)}{\leq} \mathcal{H}^{n-1}(\partial E \cap H^c)$$

Thus

$$P(E \cap H) = \mathcal{H}^{n-1}(\partial(E \cap H))$$

$$= \mathcal{H}^{n-1}(\partial E \cap H \cup \partial H \cap E)$$

$$\leq \mathcal{H}^{n-1}(\partial E \cap H) + \mathcal{H}^{n-1}(\partial H \cap E)$$

$$\overset{(1)}{\leq} \mathcal{H}^{n-1}(\partial E \cap H) + \mathcal{H}^{n-1}(\partial E \cap H^c)$$

$$= \mathcal{H}^{n-1}(\partial E) = P(E). \quad \text{ok claim!}$$

To conclude:

Fact $A \subset \mathbb{R}^n$ convex body $\Longrightarrow \exists (H_k)_{k \in \mathbb{N}}$ closed half-spaces

s.t. $A = \bigcap_{k \in \mathbb{N}} H_k$

(think of ∂A being polyhedral)

Then $A = A \cap B = B \cap \bigcap_{k=1}^{\infty} H_k$, so

$$P(A) = P\left(B \cap \bigcap_{k=1}^{\infty} H_k\right)$$

$$= P\left(\lim_{N\to+\infty} B \cap \bigcap_{k=1}^{N} H_k\right)$$

l.s.c. $\quad$ repeat claim N times

$$\leq \liminf_{N\to+\infty} P\left(B \cap \bigcap_{k=1}^{N} H_k\right)$$

Claim

$$\leq \liminf_{N\to\infty} P(B) = P(B)$$

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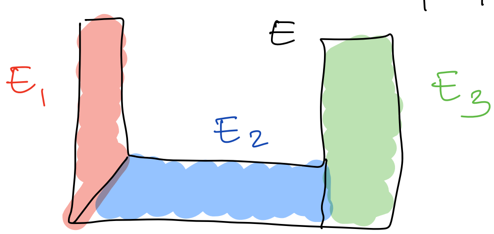
APPLICATION : CONVEX COMPONENTS

$E \subset \mathbb{R}^n$, $n \geq 2$, be just a compact set.

Decompose $E = \bigcup_{i=1}^{k} E_i$, E_i convex body,
$|E_i \cap E_j| = 0 \quad \forall i \neq j$

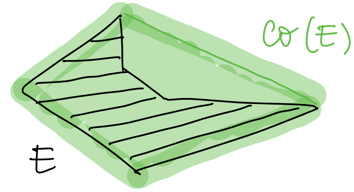


Assume a decomposition exists. What is the minimal number of convex components $k_{min}(E)$?

Remark $\quad k_{min}(E) \geq \#$ connected components

(so we can assume E is connected, $\mathrm{int}(E) \neq \emptyset$)

Consider



$co(E)$ = convex hull of E

$$= \left\{ (1-t)x + ty \; : \; x, y \in E, \; t \in [0,1] \right\} = \bigcap_{\substack{C \supset E \\ C \text{ convex}}} C \, .$$

Note that $E \subset co(E)$ and $co(E)$ is a convex body.

Assume $E = \bigcup_{i=1}^{k} E_i$, E_i convex body, $|E_i \cap E_j| = 0$.

any decomposition

Then

$$P(E) = \mathcal{H}^{n-1}\left(\partial \bigcup_{i=1}^{k} E_i \right) \leq \mathcal{H}^{n-1}\left(\bigcup_{i=1}^{k} \partial E_i \right)$$

$$\leq \sum_{i=1}^{k} \mathcal{H}^{n-1}(\partial E_i) = \sum_{i=1}^{k} P(E_i)$$

Now $E_i \subset E \subset co(E) \; \forall i \xRightarrow{\text{MONOTONICITY}} P(E_i) \leq P(co(E)) \; \forall i$

convex bodies!

Hence $P(E) \leq \sum_{i=1}^{k} P(co(E)) = k \, P(co(E))$

$$\Longrightarrow \quad k_{min}(E) \geq \left\lceil \frac{P(E)}{P(co(E))} \right\rceil$$

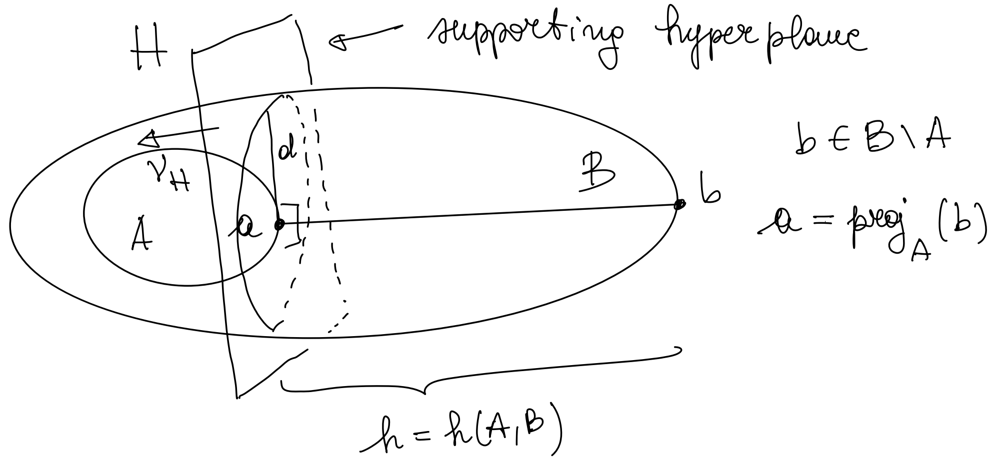
[La Civita - Leonetti, 2008]

QUANTITATIVE MONOTONICITY

Notation: $A, B \subset \mathbb{R}^n$, $n \geq 2$, convex bodies

$$h(A,B) = \sup_{b \in B} \inf_{a \in A} |a-b| = \max_{b \in B} \min_{a \in A} |a-b|.$$

is the Hausdorff distance:



$$H = \{ x \in \mathbb{R}^n : (x-a) \cdot (b-a) \leq 0 \}$$

Theorem (Carozza – Giannetti – Leonetti – Passarelli di Napoli)

- $n=2$ (2015):

$$\delta(A,B) = P(B) - P(A) \geq \frac{2h^2}{\frac{\mathcal{H}^1(B \cap \partial H)}{2} + \sqrt{\left(\frac{\mathcal{H}^1(B\cap\partial H)}{2}\right)^2 + h^2}}$$

- $n=3$ (2016): for $d = \mathrm{dist}(a, \partial B \cap \partial H)$

$$\delta(A,B) \geq \pi d h^2 \Big/ \left(d + \sqrt{d^2 + h^2} \right)$$

What about $n \geq 4$? (Levico Terme, Jan. 2016)

- (S., 2018) $\forall\, n \geq 2$

$$\delta(A,B) \geq \frac{\omega_{n-1}\, r^{n-1}\, h^2}{r + \sqrt{r^2 + h^2}}$$

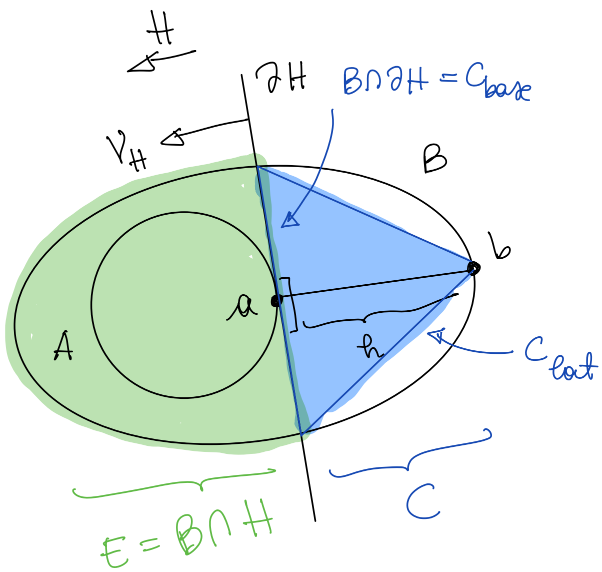
where $\quad r = \left(\dfrac{\mathcal{H}^{n-1}(B \cap \partial H)}{\omega_{n-1}} \right)^{1/n}$ (mean radius)

<u>Remark</u> $\quad n = 3 \implies r \leq d$ (stronger result)

Sketch of proof

$E = B \cap H$

C = cone from b to $B \cap \partial H$



$A \subset E \subset E \cup C \subset B$

$$\delta(A,B) = P(B) - P(A)$$

$$\geq P(E \cup C) - P(E)$$

$$P(E \cup C) = \mathcal{H}^{n-1}(\partial B \cap H) + \mathcal{H}^{n-1}(C_{lat})$$

$$P(E) = \mathcal{H}^{n-1}(\partial B \cap H) + \mathcal{H}^{n-1}(C_{base})$$

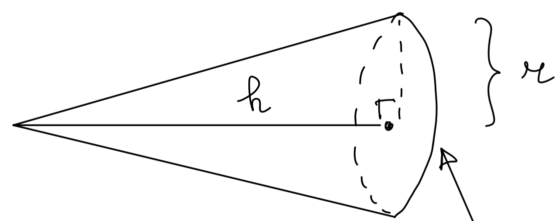
$$\Longrightarrow \quad \delta(A,B) \geq \mathcal{H}^{n-1}(C_{lat}) - \mathcal{H}^{n-1}(C_{base})$$

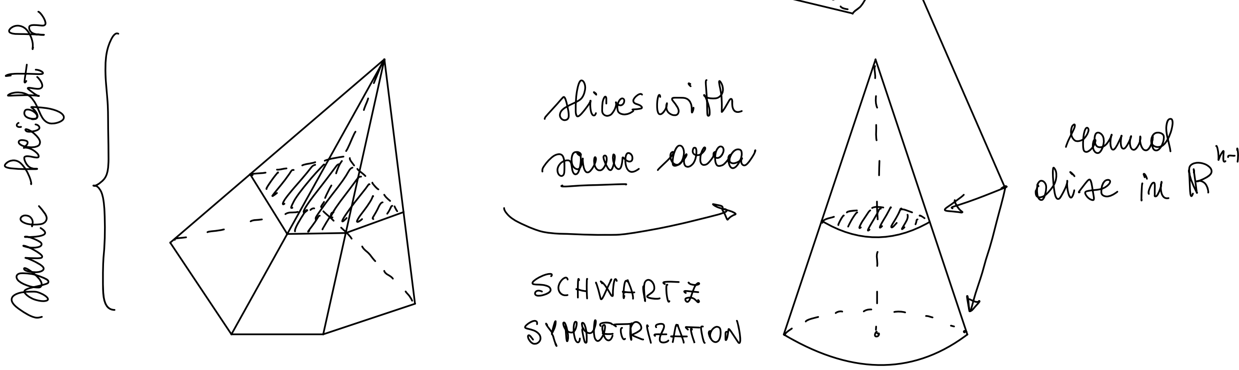
$$\mathcal{H}^{n-1}(C_{base}) = \mathcal{H}^{n-1}(B \cap \partial H) = \omega_{n-1} \underset{\substack{\uparrow \\ \text{given}}}{r}{}^{n-1}$$

Hence we need to solve

$$\inf \left\{ \mathcal{H}^{n-1}(C_{lat}) : C \text{ cone} \left\{ \begin{array}{l} \text{height} = h \\ \mathcal{H}^{n-1}(C_{base}) = \omega_{n-1} r^{n-1} \end{array} \right. \right\}$$

Using Schwartz symmetrization, the min problem is solved by the right round cone:

$C^h_r =$ 



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Application: Better lower bound on $R_{min}(E)$:

- $n=2$ C - G - L - P, 2017
- $n \geq 2$ Giannetti - S., 2022

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WEIGHTED PERIMETER

Let $f : \mathbb{R}^n \longrightarrow [0,+\infty)$ be $\mathcal{C}^1$ (WLOG).

Consider the weighted perimeter

$$P_f(E) = \int_{\partial E} f(x)\, d\mathcal{H}^{n-1}(x).$$

Theorem (Saracco – S., 2023)

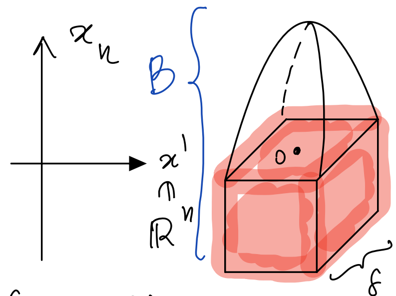
P_f monotone on convex bodies $\iff f \equiv \text{const} \geq 0$

Remark If $f = f(x,\nu)$, $\nu \in S^{n-1}$, then the theorem is FALSE! Consider $f(x,\nu) = F(x)\cdot\nu + \beta$ with

$$F \in \mathcal{C}^1, \quad \operatorname{div} F \equiv \alpha \geq 0 \quad \text{and} \quad \beta \geq \|F\|_{L^\infty}.$$

Sketch of proof

Consider:



} graph of $\lambda h(x')$ for $\lambda \geq 0$

h = concave > 0

A = small cube (size δ)

B = A $\cup$ "filled graph of λh"

Then, assuming monotonicity, $\forall \lambda \geq 0$

$$0 \leq P_f(B) - P_f(A) =$$

$$= \int_{[-\delta,\delta]^{N-1}} \underbrace{f(x',\lambda h(x'))\sqrt{1+\lambda^2|\nabla h(x')|^2}}_{\text{area of graph of } \lambda h \text{ over face of cube}} - \underbrace{f(x',0)}_{\text{area of face of cube}} dx'$$

Hence

$$\lambda \longmapsto \varphi(\lambda) = \int_{[-\delta,\delta]^{N-1}} f(x',\lambda h(x'))\sqrt{1+\lambda^2|\nabla h(x')|^2}\, dx'$$

has a minimum at $\lambda = 0$. Hence, by differentiating,

$$0 = \varphi'(0) = \int_{[-\delta,\delta]^{N-1}} \partial_{x_n} f(x',0)\, h(x')\, dx'$$

whenever $h > 0$ on $(-\delta,\delta)^{N-1}$ is concave. This forces $\partial_{x_n} f(0) = 0$. Repeating the same argument in every x_i, we get $\nabla f(0) = 0$. But the construction can be repeated at <u>any</u> point $\Longrightarrow \nabla f \equiv 0$.

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NON-LOCAL PERIMETER

"diffused" perimeter

$K : \mathbb{R}^n \longrightarrow [0,+\infty]$ measurable function

The non-local K-perimeter of E is

$$\boxed{P_K(E) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathbb{1}_E(x) - \mathbb{1}_E(y)|\, K(x-y)\, dx dy}$$

For $K(x) = \dfrac{1}{|x|^{n+s}}$, $s \in (0,1)$, we recover the s-fractional perimeter P_s of Caffarelli – Roquejoffre – Savin, 2010.

Theorem (Figalli – Fusco – Maggi – Millot – Morini, 2015
$\hookrightarrow$ fractional case
Bessas – S., 2022 (2024) & Giannetti – S., 2023)

Assume that

- $K(-x) = K(x) \quad \forall x \in \mathbb{R}^n$ (Sym)
- $\displaystyle\int_{\mathbb{R}^n} \min\{1, |x|\}\, K(x)\, dx < +\infty$ (Nt1)

(hence $P_K(E) < +\infty \quad \forall E \subset \mathbb{R}^n$ s.t. $P(E) < +\infty$)

Then

$$\underset{\text{convex bodies}}{A \subset B \subset \mathbb{R}^n} \implies P_K(A) \le P_K(B)$$

(actually in the <u>quantitative</u> sense, for K more symmetric
see Giannetti – S., 2023)

Sketch of proof

As in the classical setting, we just need to prove

Claim $\quad \begin{array}{l} E = \text{convex body} \\ H = \text{closed halfspace} \end{array} \implies P_K(E \cap H) \leq P_K(E).$

We first observe that, by (Sym),

$$P_K(E) = \int_E \int_{E^c} K(x-y)\,dx\,dy$$

Hence

$$P_K(E) - P_K(E \cap H)$$

$$= \left(\int_E \int_{E^c} - \int_{E \cap H} \int_{(E\cap H)^c} \right) K(x-y)\,dx\,dy$$

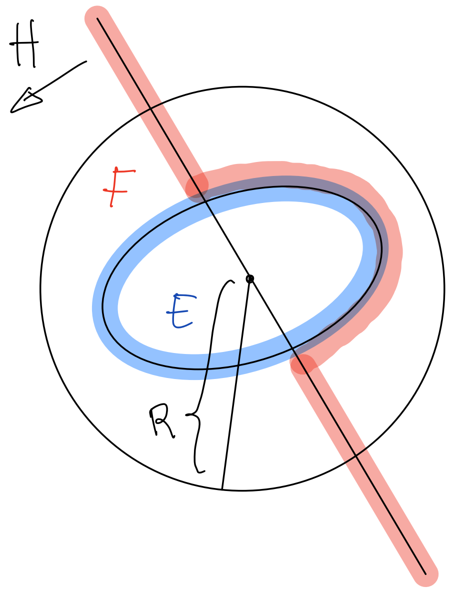
$$= \left(\int_{E \cap H} + \int_{E \setminus H} \right) \int_{E^c} K(x-y)\,dx\,dy$$

$$- \left(\int_{E^c} + \int_{E \setminus H} \right) \int_{E \cap H} K(x-y)\,dx\,dy$$

$$= \left(\int_{E^c} - \int_{E \cap H} \right) \int_{E \setminus H} K(x-y)\,dx\,dy$$

Since E is bounded, $E \subset B_R$. Define $F = E \cup H$.
Observe that

$$E \subset F, \quad E \setminus H = F \setminus H, \quad F \cap H = H, \quad F \setminus B_R = H \setminus B_R$$



Hence

$$\left(\int_{E^c} - \int_{E \cap H} \right) \int_{E \setminus H} K(x-y)\, dx dy$$

$$\geq \left(\int_{F^c} - \int_{F \cap H} \right) \int_{F \setminus H} K(x-y)\, dx dy$$

$$= P_K(F; B_R) - P_K(H; B_R)$$

where

$$P_K(E;A) = \left(\int_{E\cap A} \int_{E^c \cap A} + \int_{E \cap A} \int_{E^c \cap A^c} + \int_{E \cap A^c} \int_{E^c \cap A} \right) K(x-y)\, dx dy$$

is the K-perimeter of E inside A.

Theorem (Pagliari, 2020 & Cabré, 2020)

H is a minimum for $P_K(\cdot\,;B_R)$ $\forall R>0$, i.e.

$$P_K(H;B_R) \le P_K(E;B_R) \quad \forall E\subset\mathbb{R}^n \text{ s.t. } E\setminus B_R = H\setminus B_R.$$

LOCAL MINIMALITY OF HALF-SPACES

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Hence

$$P_K(E) - P_K(E\cap H) \overset{\text{computations}}{\ge} P_K(F;B_R) - P_K(H;B_R) \overset{\text{local minimality of } H}{\ge} 0$$

recall : $F\setminus B_R = H\setminus B_R$

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QUESTIONS

(1) When is P_f monotone for $f=f(x,\nu)$?

- $f=f(x)$: P_f monotone $\iff f\equiv \text{const}$ (Sorano – S., 2023)
- $f=g(x)\,\Phi(\nu)$ with g radial and Φ convex :

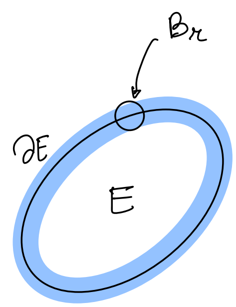
  P_f monotone $\iff g\equiv\text{const}$ (Sorano – S., 2023)

What about more general $f=f(x,\nu)$?

(2) Consider $P_r^M(E) = \dfrac{|\partial E \oplus B_r|}{2r}$ for $r>0$

Minkowski sum

Minkowski perimeter



Is P_r monotone for some $r>0$?

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