An elementary proof of existence and uniqueness for Euler and Vlasov-Poisson flows in localized Yudovich spaces

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G. Crippa and G. Stefani, 'An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces' (2021), arXiv:2110.15648.

G. Crippa, M. Inversi, C. Saffirio and G. Stefani, 'Existence and stability of weak solutions of the Vlasov-Poisson system in localized Yudovich spaces' (2023), arXiv:2306.00523.

Euler equations, velocity form

The Euler equations for an incompressible inviscid 2-dimensional fluid are given by

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \nabla p = 0 & \text{in } (0,+\infty) \times \Omega, \\ \operatorname{div} v = 0 & \text{in } [0,+\infty) \times \Omega, \\ v \cdot \nu_\Omega = 0 & \text{on } [0,+\infty) \times \partial\Omega, \\ v|_{t=0} = v_0 & \text{on } \Omega. \end{cases}$$

Objects:

- Ω is a sufficiently smooth (possibly unbounded) open set or the flat torus $\mathbb{T}^2$;
- $v \colon [0,+\infty) \times \Omega \to \mathbb{R}^2$ is the velocity of the fluid;
- $p \colon [0,+\infty) \times \Omega \to \mathbb{R}$ is the (scalar) pressure;
- $\nu_\Omega \colon \partial\Omega \to \mathbb{R}^2$ is the inner unit normal to $\partial\Omega$.

Conditions:

- $\operatorname{div} v = 0$ is the incompressibility condition;
- $v \cdot \nu_\Omega = 0$ at the boundary is the no-flow (or slip) condition.

Note: either $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{T}^2 \Rightarrow$ no boundary condition is imposed.

Euler equations, vorticity form

The vorticity $\omega\colon [0,+\infty)\times\Omega\to\mathbb{R}$ of the fluid is

$$\omega = \operatorname{curl} v$$

and satisfies

$$\begin{cases} \partial_t\omega + \operatorname{div}(v\omega) = 0 & \text{in } (0,+\infty)\times\Omega, \\ v = K\omega & \text{in } [0,+\infty)\times\Omega, \\ \omega|_{t=0} = \omega_0 & \text{on } \Omega. \end{cases}$$

Biot-Savart law: The relation $\omega = Kv$ is the Biot-Savart law, i.e.

$$v(t,x) = K\omega(t,x) = \int_\Omega k(x,y)\,\omega(t,y)\,dy,$$

where $k\colon\Omega\times\Omega\to\mathbb{R}^2$ is a convolution kernel.

Example: If $\Omega=\mathbb{R}^2$, then $k(x,y)=k_2(x-y)$ with

$$k_2(x) = \frac{1}{2\pi}\,\frac{1}{|x|^2}\begin{pmatrix}-x_2\\ x_1\end{pmatrix} = \frac{1}{2\pi}\,\frac{x^\perp}{|x|^2} \quad \text{for all } x\in\mathbb{R}^2,\ x\neq 0.$$

Literature: a quick review

Theory of strong solutions is classical (since Lichtenstein 1930).

Existence of weak solutions:

- Yudovich (1963) for $L^1 \cap L^\infty$ vorticity
- DiPerna-Majda (1987), Delort (1991), Majda (1993), Vecchi-Wu (1993), Evans-Müller (1994) for L^1 vorticity
- Serfati (1995), Vishik (1999), Taniuchi (2004) for non-decaying vorticity

Uniqueness of weak solutions:

- Yudovich (1963) for $L^1 \cap L^\infty$ vorticity
- Yudovich (1995) for unbounded vorticity with L^p-norm mildly growing
- Vishik (1999) for ∞-Besov vorticity

Philosophy: while existence follows the usual pattern

$$\text{smoothing data} \to \text{existence of smooth solutions} \to \text{compactness},$$

uniqueness is hard, due to non-linearity of Euler equations.

Warning: uniqueness is NOT expected for vorticity in L^p with $p < +\infty$!

- Vishik (2018), Albritton-Bruè-Colombo-De Lellis-Giri-Janisch-Kwon (2021)
- Bressan-Murray (2020), Bressan-Shen (2021)
- Bruè-Colombo (2021)

Properties of the kernel: less is more

Dropping time dependence, the Biot-Savart law is given by

$$v(x) = K\omega(x) = \int_\Omega k(x,y)\,\omega(y)\,dy$$

where, for some $C_1, C_2 > 0$, the kernel $k\colon \Omega \times \Omega \to \mathbb{R}^2$ satisfies

- decay: $|k(x,y)| \le \dfrac{C_1}{|x-y|}$ for all $x, y \in \Omega$, $x \ne y$;
- oscillation: $|k(x,z) - k(y,z)| \le C_2 \dfrac{|x-y|}{|x-z|\,|y-z|}$ for all $x, y, z \in \Omega$, $z \ne x, y$.

From the relation $v = K\omega$, we also get

- incompressibility: $\operatorname{div}(K\omega) = 0$;
- no-flow: $(K\omega) \cdot \nu_\Omega = 0$ at the boundary.

IDEA: try to rely on the above 'metric' properties of k only!

A posteriori: we can even relax the incompressibility property to

- controlled compression: $\|\operatorname{div}(K\omega)\|_{L^\infty(\Omega)} \le C_3 \|\omega\|_{L^1(\Omega)}$ for some $C_3 > 0$.

Fix $x, y \in \Omega$ with $d = |x-y| < 1$. We can split

$$|K\omega(x) - K\omega(y)| \leq \int_\Omega |k(x,z) - k(y,z)|\,|\omega(z)|\,dz$$
$$= \left(\int_{\Omega \setminus B_2(x)} + \int_{\Omega \cap (B_2(x) \setminus B_{2d}(x))} + \int_{\Omega \cap B_{2d}(x)} \right) |k(x,z) - k(y,z)|\,|\omega(z)|\,dz.$$

We can estimate

$$\int_{\Omega \setminus B_2(x)} \cdots \overset{\text{oscillation}}{\lesssim} |x-y| \int_{\Omega \setminus B_2(x)} \frac{|\omega(z)|}{|x-z|\,|y-z|}\,dz \lesssim |x-y|\,\|\omega\|_{L^1(\Omega)}$$

$$\int_{\Omega \cap (B_2(x) \setminus B_{2d}(x))} \cdots \overset{\text{oscillation}}{\lesssim} \int_{\Omega \cap (B_2(x) \setminus B_{2d}(x))} \frac{|\omega(z)|}{|x-z|^2}\,dz$$

$$\int_{\Omega \cap B_{2d}(x)} \cdots \overset{\text{decay}}{\lesssim} \int_{\Omega \cap B_{2d}(x)} \frac{|\omega(z)|}{|x-z|}\,dz + \int_{\Omega \cap B_{3d}(y)} \frac{|\omega(z)|}{|y-z|}\,dz$$

We need to control

$$\alpha(d) = \sup_{x\in\Omega} \int_{\Omega\cap(B_2(x)\setminus B_{2d}(x))} \frac{|\omega(z)|}{|x-z|^2}\,dz \quad \text{and} \quad \beta(d) = \sup_{x\in\Omega} \int_{\Omega\cap B_{3d}(x)} \frac{|\omega(z)|}{|x-z|}\,dz$$

defined for $d \in (0,1]$. By Hölder's inequality, we have

$$\begin{aligned}\alpha(d) &\lesssim \left(\sup_{x\in\Omega} \|\omega\|_{L^p(\Omega\cap B_2(x))}\right) \left(\int_{2d}^{2} r^{1-2p'}\,dr\right)^{1/p'} \\ &\lesssim C \left(\frac{2^{2-2p'}}{2p'-2}\right)^{1/p'} \left(d^{2-2p'}-1\right)^{1/p'} \lesssim C\,p\,d^{-2/p}\end{aligned}$$

and, similarly,

$$\begin{aligned}\beta(d) &\lesssim \left(\sup_{x\in\Omega} \|\omega\|_{L^p(\Omega\cap B_3(x))}\right) \left(\int_{0}^{3d} r^{1-p'}\,dr\right)^{1/p'} \\ &\lesssim C \left(\frac{3^{2-p'}}{2-p'}\right)^{1/p'} d^{(2-p')/p'} \lesssim C\,\frac{p}{p-2}\,d^{1-2/p},\end{aligned}$$

where $C = \sup_{x\in\Omega} \|\omega\|_{L^p(\Omega\cap B_1(x))}$.

We let

$$L^p_{\mathrm{ul}}(\Omega) = \left\{ f \in L^p_{\mathrm{loc}}(\Omega) \ : \ \|f\|_{L^p_{\mathrm{ul}}(\Omega)} = \sup_{x\in\Omega} \|f\|_{L^p(\Omega\cap B_1(x))} < +\infty \right\}$$

be the uniformly-localized L^p space on Ω. Note that radius $= 1$ is not restrictive.

Theorem (Hölder continuity)

Let $p \in (2, +\infty)$. If $\omega \in L^1(\Omega) \cap L^p_{\mathrm{ul}}(\Omega)$, then $K\omega \in C_b^{0,1-2/p}(\Omega;\mathbb{R}^2)$ with

$$\|K\omega\|_{L^\infty(\Omega;\mathbb{R}^2)} \lesssim \max\left\{1, \tfrac{1}{p-2}\right\} \left(\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L^p_{\mathrm{ul}}(\Omega)}\right)$$

$$|K\omega(x) - K\omega(y)| \lesssim \max\left\{1, \tfrac{1}{p-2}\right\} \left(\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L^p_{\mathrm{ul}}(\Omega)}\right) p\, |x-y|^{1-2/p} \quad \forall x, y \in \Omega.$$

Remark: the result is not a surprise, since (for the Biot-Savart kernel)

CZ theory + Morrey's inequality $\Rightarrow$ Hölder continuity.

However, our proof is surprising elementary!

We let

$$Y^{\Theta}_{\mathrm{ul}}(\Omega) = \left\{ f \in \bigcap_{p\in[1,+\infty)} L^p_{\mathrm{ul}}(\Omega) \ : \ \|f\|_{Y^{\Theta}_{\mathrm{ul}}(\Omega)} = \sup_{p\in[1,+\infty)} \frac{\|f\|_{L^p_{\mathrm{ul}}(\Omega)}}{\Theta(p)} < +\infty \right\}$$

be the uniformly-localized Yudovich space on Ω associated to Θ.

If $\omega \in L^1(\Omega) \cap Y^{\Theta}_{\mathrm{ul}}(\Omega)$, then for all $p \ge 3$ we have

$$\begin{aligned} |K\omega(x) - K\omega(y)| &\lesssim \max\left\{1, \tfrac{1}{p-2}\right\} (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L^p_{\mathrm{ul}}(\Omega)})\, p\, |x-y|^{1-2/p} \\ &\lesssim (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y^{\Theta}_{\mathrm{ul}}(\Omega)})\, \Theta(p)\, p\, |x-y|^{1-2/p}. \end{aligned}$$

If $d = |x-y| \ll 1$, then we can take $p = |\log d| \gg 1$ and observe that

$$\Theta(p)\, p\, |x-y|^{1-2/p} = \Theta(|\log d|)\, |\log d|\, d^{1-\frac{2}{|\log d|}} \eqsim d\, |\log d|\, \Theta(|\log d|)$$

since $d^{-\frac{2}{|\log d|}} = \exp(\frac{2}{\log d} \cdot \log d) = e^2$.

Modulus of continuity φ_Θ

We let the function $\varphi_\Theta \colon [0,+\infty) \to [0,+\infty)$ be such that $\varphi_\Theta(0)=0$ and

$$\varphi_\Theta(r) = \begin{cases} r\,(1-\log r)\,\Theta(1-\log r) & \text{for } r \in (0, e^{-2}] \\ e^{-2}\,3\,\Theta(3) & \text{for } r > e^{-2}. \end{cases}$$

We say that φ_Θ is the modulus of continuity associated to Θ and define

$$C_b^{0,\varphi_\Theta}(\Omega;\mathbb{R}^2) = \left\{ v \in L^\infty(\Omega;\mathbb{R}^2) \ : \ \sup_{x\neq y} \frac{|v(x)-v(y)|}{\varphi_\Theta(|x-y|)} < +\infty \right\}.$$

Corollary (φ_Θ-continuity)

If $\omega \in L^1(\Omega) \cap Y_{\mathrm{ul}}^\Theta(\Omega)$, then $K\omega \in C_b^{0,\varphi_\Theta}(\Omega;\mathbb{R}^2)$ with

$$\|K\omega\|_{L^\infty(\Omega;\mathbb{R}^2)} \lesssim \|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y_{\mathrm{ul}}^\Theta(\Omega)}$$

$$|K\omega(x) - K\omega(y)| \lesssim \left(\|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y_{\mathrm{ul}}^\Theta(\Omega)}\right) \varphi_\Theta(|x-y|) \quad \forall x,y \in \Omega.$$

Remark: we recover Yudovich's continuity modulus, with NO sharp tools!

By definition of φ_Θ, note that
$$\int_{0+} \frac{dr}{\varphi_\Theta(r)} = \int^{+\infty} \frac{dp}{p\,\Theta(p)}.$$

Theorem (Existence)

Let $p > 2$. Given $\omega_0 \in L^1(\Omega) \cap L^p_{\mathrm{ul}}(\Omega)$, there is a weak sol. (ω, v) of (E) such that
$$\omega \in L^\infty_{\mathrm{loc}}([0,+\infty); L^1(\Omega) \cap L^p_{\mathrm{ul}}(\Omega)), \quad v \in L^\infty_{\mathrm{loc}}([0,+\infty); C_b^{0,1-2/p}(\Omega;\mathbb{R}^2)).$$
Moreover, if $\omega_0 \in L^1(\Omega) \cap Y^\Theta_{\mathrm{ul}}(\Omega)$, then (ω, v) is such that
$$\omega \in L^\infty_{\mathrm{loc}}([0,+\infty); L^1(\Omega) \cap Y^\Theta_{\mathrm{ul}}(\Omega)), \quad v \in L^\infty_{\mathrm{loc}}([0,+\infty); C_b^{0,\varphi_\Theta}(\Omega;\mathbb{R}^2))$$
and, provided that φ_Θ is Osgood, (ω, v) is Lagrangian.

ODE theory: φ_Θ Osgood $\Rightarrow$ there is a unique flow X such that $\frac{d}{dt}X(t,\cdot) = v(t,X)$.

Lagrangian: the solution is such that $\omega(t,\cdot) = X(t,\cdot)_\# \omega_0$ (push-forward).

Remark: it applies to Θ not BMO (e.g., $\Theta(p) \approx p^\alpha$) and to non-Biot-Savart kernels.

Strategy of proof for existence

Warning: we cannot rely on the existence of smooth solutions!

Indeed, the kernel is general, so there are NO equations in velocity form.

We have to follow a different strategy:

1) construct a solution in $L^1 \cap L^\infty$ via time-stepping argument;
2) construct a solution in $L^1 \cap L^p_{\text{ul}}$ by truncating the initial data;
3) show that the construction preserves the $L^1 \cap Y^\Theta_{\text{ul}}$-regularity.

To gain existence, we need a compactness criterion à la Aubin-Lions:

- the proof exploits the Dunford-Pettis, Lusin and Arzelà-Ascoli Theorems;
- we assume weak compactness, while usually one takes strong compactness.

Compactness criterion

Theorem (Baby Aubin-Lions)

Let $T > 0$ and let $(f^n)_{n\in\mathbb{N}} \subset L^\infty([0,T]; L^1(\Omega))$ be a bounded sequence which is equi-integrable in space uniformly in time:

- $\sup_{n\in\mathbb{N}} \|f^n\|_{L^\infty([0,T];\, L^1(\Omega))} < +\infty$
- $\forall \varepsilon > 0\ \exists \delta > 0 \,:\, A \subset \Omega,\ |A| < \delta \Rightarrow \sup_{n\in\mathbb{N}} \|f^n\|_{L^\infty([0,T];\, L^1(A))} < \varepsilon$
- $\forall \varepsilon > 0\ \exists \Omega_\varepsilon \subset \Omega$ with $|\Omega_\varepsilon| < +\infty \,:\, \sup_{n\in\mathbb{N}} \|f^n\|_{L^\infty([0,T];\, L^1(\Omega\setminus\Omega_\varepsilon))} < \varepsilon$.

Assume that, for each $\varphi \in C_c^\infty(\Omega)$, the functions $F_n[\varphi] \colon [0,T] \to \mathbb{R}$, given by

$$F_n[\varphi](t) = \int_\Omega f^n(t,\cdot)\,\varphi\,dx, \quad t \in [0,T],$$

are uniformly equi-continuous on $[0,T]$.

Then there exist a subsequence $(f^{n_k})_{k\in\mathbb{N}}$ and $f \in L^\infty([0,T]; L^1(\Omega))$ such that

$$\lim_{k\to+\infty} \int_\Omega f^{n_k}(t,\cdot)\,\varphi\,dx = \int_\Omega f(t,\cdot)\,\varphi\,dx$$

for a.e. $t \in [0,T]$ and all $\varphi \in L^\infty(\Omega)$.

Uniqueness

Theorem (Uniqueness)

Let Θ be such that φ_Θ is concave and Osgood. There is at most one (Lagrangian) weak solution (ω, v) of (E) such that

$$\omega \in L^\infty_{\text{loc}}([0,+\infty); L^1(\Omega) \cap Y^\Theta_{\text{ul}}(\Omega)), \quad v \in L^\infty_{\text{loc}}([0,+\infty); C_b^{0,\varphi_\Theta}(\Omega;\mathbb{R}^2)),$$

starting from $\omega_0 \in L^1(\Omega) \cap Y^\Theta_{\text{ul}}(\Omega)$, $v_0 = K\omega_0$.

Remark: our uniqueness result

- recovers (and actually improves) Yudovich's uniqueness theorem;
- is proved in a Lagrangian way, we do not use the energy method;
- does not rely on the specific structure of the Biot-Savart kernel.

Careful: Osgood velocity $\Rightarrow$ any weak solution is Lagrangian, but this is delicate!

- Ambrosio-Bernard (2008) via superposition principle
- Caravenna-Crippa (2021) via integral curves
- Clop-Jylhä-Mateu-Orotobig (2019) via optimal transport

Vlasov-Poisson (generalized) equations

Fix an antisymmetric kernel k (usually $k(x) = \kappa \dfrac{x}{|x|^d}$ with $\kappa = \pm 1$) and consider

$$\begin{cases} \partial_t f + F \cdot \nabla_x f + E_f \cdot \nabla_v f = 0 & \text{in } (0,T) \times \mathbb{R}^{2d}, \\ E_f(t,x) = \displaystyle\int_{\mathbb{R}^d} k(x,y)\, \varrho_f(t,y)\, dy & \text{for } t \in [0,T],\ x \in \mathbb{R}^d, \\ \varrho_f(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, dv & \text{for } t \in [0,T],\ x \in \mathbb{R}^d, \\ f(0,\cdot) = f_0 & \text{on } \mathbb{R}^{2d}, \end{cases}$$

where $f \in L^\infty([0,T]; L^1(\mathbb{R}^{2d}))$, $f_0 \in L^1(\mathbb{R}^{2d})$ and $F \in L^\infty([0,T]; C(\mathbb{R}^{2d}; \mathbb{R}^d))$ is such that $\operatorname{div}_x F = 0$ and, for some $L \geq 0$,

$$\sup_{t \in [0,T]} |F(t,x,v) - F(t,y,w)| \leq L\,[|x-y| + |v-w|] \quad \forall x,y,v,w \in \mathbb{R}^d.$$

Cases: $F(t,x,v) = v$ is classical, while $F(t,x,v) = \dfrac{v}{\sqrt{1+|v|^2}}$ is relativistic.

Meaning: the time evolution of the density f of plasma consisting of charged particles with long-range interaction (repulsive for $\kappa = 1$, attractive for $\kappa = -1$).

Admissible densities and modulus of continuity φ_Θ

We work in the class of admissible densities

$$\mathcal{A}^\Theta([0,T]) = \{ f \in L^\infty([0,T]; L^1(\mathbb{R}^{2d})) : \varrho_f \in L^\infty([0,T]; Y^\Theta_{\mathrm{ul}}(\mathbb{R}^d)) \}$$

for some fixed increasing growth function $\Theta \colon [0,+\infty) \to (0,+\infty)$. Examples:

- $\Theta(p) = c > 0$ by Loeper (2006);
- $\Theta(p) = p$ by Miot (2016);
- $\Theta(p) = p^{\frac{1}{\alpha}}$ for $\alpha \in [1,\infty)$ by Holding-Miot (2018).

We want to study the regularity of the ('electric') vector field

$$E_f(t,x) = K\varrho_f = \int_{\mathbb{R}^d} k(x,y)\,\varrho_f(t,y)\,dy \quad \text{for } f \in \mathcal{A}^\Theta([0,T]).$$

We define the modulus of continuity associated to Θ as

$$\varphi_\Theta(r) = \begin{cases} r\,|\log r|\,\Theta(|\log r|) & \text{for } r \in [0, e^{-d-1}), \\ e^{-d-1}\,(d+1)\,\Theta(d+1) & \text{for } r \in [e^{-d-1}, +\infty). \end{cases}$$

Proposition (φ_Θ-continuity)

If $f \in \mathcal{A}^\Theta([0,T])$, then $E_f \in L^\infty([0,T]; C_b^{0,\varphi_\Theta}(\mathbb{R}^d;\mathbb{R}^d))$.

Remark: we recover regularity in [Loeper], [Miot] and [Holding-Miot] elementarily!

Wasserstein stability

The 1-Wasserstein distance between $f_1\mathscr{L}^d, f_2\mathscr{L}^d \in \mathscr{P}_1(\mathbb{R}^{2d})$ is given by

$$\mathsf{W}_1(f_1,f_2) = \sup\left\{\int_{\mathbb{R}^{2d}} \psi(f_1-f_2)\,d\mathscr{L}^{2d} : \psi\in \mathrm{Lip}(\mathbb{R}^{2d}),\ \mathrm{Lip}(\psi)\le 1\right\}.$$

Assume that the primitive $\Phi_\Theta(r) = \int_0^r \varphi_\Theta(s)\,ds$ satisfies $\int_{0+} \frac{dr}{\sqrt{\Phi_\Theta(r)}} = \infty$.

Theorem (Lagrangian stability)

If $f_1, f_2 \in \mathcal{A}^\Theta([0,T];\mathscr{P}_1(\mathbb{R}^{2d}))$ are two Lagrangian solution relative to (F_1,E_1), $E_1 = E_{f_1}$, and (F_1,E_1), $E_2 = E_{f_2}$, with initial datum $f_0^1, f_0^2 \in \mathscr{P}_1(\mathbb{R}^{2d})$, then

$$\sup_{t\in[0,T]} \mathsf{W}_1(f_1(t,\cdot), f_2(t,\cdot)) \le \Omega_\Theta\big(\mathsf{W}_1(f_0^1,f_0^2), \|F_1-F_2\|_{L^\infty}\big)$$

Uniqueness: in particular, if $f_0^1 = f_0^2$ and $F_1 = F_2$, then $f_1 = f_2$.

Lagrangian: $f(t,\cdot) = (X,V)(t,\cdot)_\# f_0$, where $(X,V)(t,\cdot)$ solves the ODE

$$\begin{cases}\dot X = F(t,X,V)\\ \dot V = E_f(t,X)\end{cases} \qquad \text{with } X(0)=x,\ V(0)=v.$$

Why an Osgood condition on the primitive of φ_Θ?

Schematically, we are dealing with an ODE of the form

$$\begin{cases} \dot{X} = F(t, X, V) \\ \dot{V} = E(t, X) \end{cases} \qquad \text{with } X(0) = x,\ V(0) = v,$$

where $F \in \mathrm{Lip}_b$ and $E \in C_b^{0,\varphi_\Theta}$.

Assume $F(t, X, V) = V$ and $E(t, X) = \varphi_\Theta(X)$ for simplicity. Then

$$\begin{cases} \dot{X} = V \\ \dot{V} = \varphi_\Theta(X) \end{cases} \qquad \text{with } X(0) = x,\ V(0) = v,$$

which is a 2nd order problem!

Assume $d = 1$ and $X(0) = V(0) = 0$ for simplicity. Then $\frac{d}{dt}\frac{\dot{X}^2}{2} = \varphi_\Theta(X)\,\dot{X}$ and

$$\dot{X}^2(t) = 2\int_0^t \varphi_\Theta(X(s))\,\dot{X}(s)\,ds = 2\Phi_\Theta(X(t)).$$

Hence uniqueness for the ODE requires

$$\int_{0+} \frac{\dot{X}(t)\,dt}{\sqrt{\Phi_\Theta(X(t))}} = \int_{0+} \frac{dr}{\sqrt{\Phi_\Theta(r)}} = \infty.$$

Existence of Lagrangian admissible solutions

We just work with $k(x) = \pm \dfrac{x}{|x|^d}$ and $d = 2, 3$.

Theorem (Existence)

If $\vartheta \in Y^{\Theta}(\mathbb{R}^d)$ satisfies

$$\vartheta \not\equiv 0, \quad \vartheta \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \vee |x|)\, \vartheta(x)\, dx < +\infty,$$

then there is a Lagrangian solution $f \in \mathcal{A}^{\Theta}([0,T])$ starting from the initial datum

$$f_0(x,v) = \frac{1_{(-\infty,0]}\left(|v|^2 - \vartheta(x)^{\frac{2}{d}}\right)}{|B_1|\, \|\vartheta\|_{L^1}}, \quad \text{for } x, v \in \mathbb{R}^d,$$

such that $f(t,\cdot)\,\mathscr{L}^{2d} \in \mathscr{P}_1(\mathbb{R}^{2d})$ for all $t \in [0,T]$ and

$$C\,\|\vartheta\|_{L^p} \leq \|\varrho_f\|_{L^\infty([0,T];L^p)} \leq C_T \|\vartheta\|_{L^p} \quad \text{for all } p \in [1,+\infty)\,,$$

for some constants $C, C_T > 0$, where C_T depends on T.

Remark: the result is based upon the deep existence theorem by [Lions-Perthame].

Am I cheating about existence?

Note that we start with $\vartheta \in Y^{\Theta}(\mathbb{R}^d)$ such that

$$\vartheta \not\equiv 0, \quad \vartheta \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \vee |x|)\, \vartheta(x)\, dx < +\infty,$$

and find a Lagrangian admissible solution with

$$C\, \|\vartheta\|_{L^p} \leq \|\varrho_f\|_{L^\infty([0,T];L^p)} \leq C_T \|\vartheta\|_{L^p} \quad \text{for all } p \in [1, +\infty)\,.$$

Achtung! Any (non-zero) $0 \leq \vartheta \in C_c(\mathbb{R}^d)$ meets the requirements!

Hence the existence becomes truly interesting if ϑ also satisfies $\inf_{p\geq 1} \dfrac{\|\vartheta\|_{L^p}}{\Theta(p)} > 0$.

We have non-trivial existence for any Θ_m given by

$$\Theta_m(p) = p\, \log(p)^2\, \log\log(p)^2 \cdots \underbrace{\log\log\cdots\log}_{m\ \text{times}}(p)^2.$$

Proposition (Saturation of Θ_m)

For each $m \geq 0$, Φ_{Θ_m} satisfies the Osgood condition and there is $\vartheta_m \in Y^{\Theta_m}(\mathbb{R}^d)$ with compact support satisfying all the requirements above.

Remark: we recover the existence results in [Miot] and [Holding-Miot].

Futurama

Project 1: for 2D Euler equations remove the L^1 assumption, dealing with weak solutions in Y_{ul}^{Θ} for suitable Θ, in collaboration with G. Ciampa and G. Crippa.

Project 2: for Vlasov-Poisson (generalized) system, prove that the Lagrangian assumption is not needed à la Ambrosio-Bernard, in collaboration with M. Inversi.

Other ideas: more general functional spaces? other equations?

Thank you for your attention!

G. Crippa and G. Stefani, An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces (2021), arXiv:2110.15648.

G. Crippa, M. Inversi, C. Saffirio and G. Stefani, Existence and stability of weak solutions of the Vlasov-Poisson system in localized Yudovich spaces (2023), arXiv:2306.00523.

Slides available on my webpage: giorgiostefani.weebly.com

Assume (ω^1, v^1) and (ω^2, v^2) are two Lagrangian solutions with same initial datum.

We can thus write $\omega^i = X^i(t,\cdot)_{\#}\omega_0$ where X^i is the flow associated to v^i, $i = 1, 2$.

Fix $T > 0$ and consider $t \in [0, T]$. We start with the usual splitting

$$\begin{aligned} |X^1 - X^2| &\leq \int_0^t |v^1(s, X^1) - v^2(s, X^2)|\, ds \\ &\leq \int_0^t |v^1(s, X^1) - v^1(s, X^2)|\, ds + \int_0^t |v^1(s, X^2) - v^2(s, X^2)|\, ds. \end{aligned}$$

The first term is easy, we can use the φ_Θ-continuity and obtain

$$|v^1(s, X^1) - v^1(s, X^2)| \lesssim \varphi_\Theta(|X^1 - X^2|),$$

with implicit constant depending on $\|\omega^1\|_{L^\infty([0,T]; L^1 \cap Y^\Theta_{\mathrm{ul}})}$.

Proof of uniqueness 2/4

The second term is delicate. We use $v = K\omega$ and the push-forward to get

$$\begin{aligned}
|v^1(s,X^2) - v^2(s,X^2)| &= |(K\omega^1)(s,X^2) - (K\omega^2)(s,X^2)| \\
&= \left| \int_\Omega k(X^2,y)\,\omega^1(s,y)\,dy - \int_\Omega k(X^2,y)\,\omega^2(s,y)\,dy \right| \\
&= \left| \int_\Omega k(X^2,X^1(s,y))\,\omega_0(y)\,dy - \int_\Omega k(X^2,X^2(s,y))\,\omega_0(y)\,dy \right| \\
&\le \int_\Omega |k(X^2,X^1(s,y)) - k(X^2,X^2(s,y))|\,|\omega_0(y)|\,dy.
\end{aligned}$$

We combine the two estimates and obtain

$$\begin{aligned}
|X^1 - X^2| \le \int_0^t \varphi_\Theta(|X^1 - X^2|)\,dt \\
+ \int_0^t \int_\Omega |k(X^2,X^1(s,y)) - k(X^2,X^2(s,y))|\,|\omega_0(y)|\,dy\,dt.
\end{aligned}$$

Now choose the finite measure $\mu = \bar\omega\,\mathscr{L}^2$, with $\bar\omega = |\omega_0| + \eta$ and $0 < \eta \in L^1 \cap L^\infty$.

Proof of uniqueness 3/4

We integrate with respect to μ. By Tonelli Theorem, we can estimate

$$
\begin{aligned}
&\int_\Omega \int_\Omega |k(X^2(s,x), X^1(s,y)) - k(X^2(s,x), X^2(s,y))|\, |\omega_0(y)|\, dy\, d\mu(x) \\
&= \int_\Omega |\omega_0(y)| \int_\Omega |k(X^2(s,x), X^1(s,y)) - k(X^2(s,x), X^2(s,y))|\, d\mu(x)\, dy \\
&= \int_\Omega |\omega_0(y)| \int_\Omega |k(x, X^1(s,y)) - k(x, X^2(s,y))|\, X^2(s,\cdot)_\# \bar{\omega}(x)\, dx\, dy \\
&\overset{(!)}{\lesssim} \int_\Omega |\omega_0(y)|\, \varphi_\Theta(|X^1(s,y) - X^2(s,y)|)\, dy \\
&\le \int_\Omega \varphi_\Theta(|X^1(s,y) - X^2(s,y)|)\, d\mu(y).
\end{aligned}
$$

Inequality (!) follows from the same computations for the φ_Θ-continuity of velocity.

The implicit constant depends on $\|\bar{\omega}\|_{L^\infty([0,T];L^1 \cap Y^\Theta_{\mathrm{ul}})}$. But $\bar{\omega} = |\omega_0| + \eta$, so we can choose $\eta \in L^1 \cap L^\infty$ to let the constant depend on $\|\omega_0\|_{L^\infty([0,T];L^1 \cap Y^\Theta_{\mathrm{ul}})}$ only!

Proof of uniqueness 4/4

In conclusion, we get

$$\int_\Omega |X^1 - X^2|\,d\mu \lesssim \int_0^t \int_\Omega \varphi_\Theta(|X^1 - X^2|)\,d\mu\,dt.$$

But φ_Θ is concave and Osgood, so that

$$\int_\Omega \varphi_\Theta(|X^1 - X^2|)\,d\mu \overset{\text{Young}}{\leq} \varphi_\Theta\left(\int_\Omega |X^1 - X^2|\,d\mu\right)$$

and thus

$$\xi(t) \leq \int_0^t \varphi_\Theta(\xi(s))\,dt, \qquad \xi(s) = \int_\Omega |X^1(s,\cdot) - X^2(s,\cdot)|\,d\mu,$$

imply that $X^1 = X^2$ for all $t \in [0,T]$, which means $\omega^1 = \omega^2$ and so $v^1 = v^2$.