Double Hölder regularity of the hydrodynamic pressure in bounded domains

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[1] L. De Rosa, M. Latocca and GS, On double Hölder regularity of the hydrodynamic pressure in bounded domains, Calc. Var. PDEs 62 (2023), no. 3, Paper No. 85, 31.

[2] L. De Rosa, M. Latocca and GS, Full double Hoölder regularity of the pressure in bounded domains (2023), submitted, preprint available at arXiv:2301.06482.

Euler equations and the hydrodynamic pressure

The evolution of an incompressible inviscid fluid is described by the Euler equations

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 & \text{in } \Omega \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,T), \\ u \cdot n = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{E}$$

where:

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded and simply connected C^2 domain;
- $u \colon \Omega \times (0,T) \to \mathbb{R}^d$ is the velocity;
- $p \colon \Omega \times (0,T) \to \mathbb{R}$ is the hydrodynamic pressure;
- $n \colon \partial\Omega \to \mathbb{R}^d$ is the outward unit normal to $\partial\Omega$;
- $u(\cdot,t) \cdot n = 0$ on $\partial\Omega$ is the usual no-flow condition.

Hence, at each time, the pressure p solves the elliptic Neumann problem

$$\begin{cases} -\Delta p = \operatorname{div}\operatorname{div}(u \otimes u) & \text{in } \Omega, \\ \partial_n p = u \otimes u : \nabla n & \text{on } \partial\Omega, \end{cases} \tag{P}$$

noticing that

$$\partial_n p = \nabla p \cdot n = -\operatorname{div}(u \otimes u) \cdot n = -u \cdot \nabla(u \cdot n) + u \otimes u : \nabla n = u \otimes u : \nabla n.$$

Regularity problem for the pressure

Assume $u \in C^{0,\gamma}(\Omega)$ with $\operatorname{div} u = 0$ and $u \cdot n = 0$ on $\partial\Omega$ and consider

$$\begin{cases} -\Delta p = \operatorname{div}\operatorname{div}(u \otimes u) & \text{in } \Omega, \\ \partial_n p = u \otimes u : \nabla n & \text{on } \partial\Omega. \end{cases} \tag{P}$$

By standard Schauder's estimates $p \in C^{0,\gamma}_{\text{loc}}(\Omega)$, but can we do better?

The regularity of the pressure doubles the regularity of the velocity!

Theorem (Silvestre, 2011; Isett, 2023; Constantin, 2014; Colombo-De Rosa, 2020)

If $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, then $u \in C^{0,\gamma} \implies p \in \begin{cases} C^{0,2\gamma} & \text{if } 0 < \gamma < \frac{1}{2}, \\ \text{Lip}_{\log} & \text{if } \gamma = \frac{1}{2}, \\ C^{1,2\gamma-1} & \text{if } \frac{1}{2} < \gamma < 1. \end{cases}$

Notation: $p \in \text{Lip}_{\log}$ means $|p(x) - p(y)| \leq C|x-y| \min\{1, |\log|x-y||\}$.

Remark: See [Colombo-De Rosa-Forcella, 2020] for Sobolev and Besov spaces.

Applications: smoothness of trajectories of Euler flows, intermittency phenomena in turbulent flows, anomalous dissipation...

Idea of proof in $\mathbb{R}^d$ (after Colombo-De Rosa)

The solution of $-\Delta p = \operatorname{div}\operatorname{div} M$ decaying at infinity can be written as

$$p(x) = \int_{B_R(\bar{x})} \partial^2_{ij} G^D_{\mathbb{R}^d}(x-y)(M^{ij}(y) - M^{ij}(x))\,dy$$
$$- M^{ij}(x) \int_{\partial B_R(\bar{x})} \partial_i G^D_{\mathbb{R}^d}(x-y)\nu_j(y)\,dy$$

via Green-Dirichlet kernel $G^D_{\mathbb{R}^d}$, where $M \in C^{0,\gamma}_c(\mathbb{R}^d;\mathbb{R}^{d\times d})$ with $\operatorname{supp} M \subset B_R(x_0)$.

Fix $x_1, x_2 \in \mathbb{R}^d$ and $\bar{x} = \frac{x_1+x_2}{2}$ and note that

$$\operatorname{div}\operatorname{div}(u\otimes u) = (\partial_i u^i)(\partial_j u^j) = \partial_i(u^i - u^i(x_1))\,\partial_j(u^j - u^j(x_2))$$
$$= \partial^2_{ij}\big((u^i - u^i(x_1))\,(u^j - u^j(x_2))\big),$$

so that we can write (for $\gamma > \frac{1}{2}$ one takes derivatives $\partial_k p$ and $\partial^3_{ijk} G^D_{\mathbb{R}^d}$)

$$p(x_1) - p(x_2)$$
$$= \int_{B_R(\bar{x})} \big[\partial^2_{ij} G^D_{\mathbb{R}^d}(x_1 - y) - \partial^2_{ij} G^D_{\mathbb{R}^d}(x_2 - y)\big](u^i(y) - u^i(x_1))\,(u^j(y) - u^j(x_2))\,dy$$

and the conclusion follows by the decay properties of (derivatives of) $G^D_{\mathbb{R}^d}$.

Pressure regularity on bounded domains

In the presence of a boundary, even the single regularity of the pressure is hard!

Theorem (Bardos-Titi, 2022; Bardos-Boudros-Titi, 2023)

$$d = 2 \text{ and } \partial\Omega \in C^2 \quad \text{or} \quad d = 3 \text{ and } \partial\Omega \in C^3 \implies p \in C^{0,\gamma}(\Omega)$$

Question: Can we still double the regularity on bounded sets?

Theorem #1 (De Rosa-Latocca-S, 2023)

$$d \geq 2 \text{ and } \partial\Omega \in C^{2,\delta} \implies p \in \begin{cases} C^{0,\gamma} & \text{if } 0 < \gamma \leq \frac{1}{2}, \\ C^{1,\min\{\delta, 2\gamma-1\}} & \text{if } \frac{1}{2} < \gamma < 1. \end{cases}$$

By interpolation, if $\gamma < \frac{1}{2}$ then $p \in C^{0,2\gamma-\varepsilon}$ for $\varepsilon \in (0, 2\gamma)$ with $d \geq 2$ and $\partial\Omega \in C^{3,\delta}$.

Theorem #2 (De Rosa-Latocca-S, 2023)

$$d \geq 2 \text{ and } \partial\Omega \in C^{2,1} \implies p \in \begin{cases} C^{0,2\gamma} & \text{if } 0 < \gamma < \frac{1}{2}, \\ C^1_* & \text{if } \gamma = \frac{1}{2}. \end{cases}$$

Notation: $C^1_* \subsetneq \text{Lip}_{\log}$ with $\|p\|_{C^1_*} = \|p\|_{L^\infty} + \sup_{x,h \in \mathbb{R}^d,\, h \neq 0} \frac{|p(x+h)+p(x-h)-2p(x)|}{|h|}$.

Strategy of the proof of Theorem #1

Assume $\Omega \subset \mathbb{R}^d$ is a simply connected bounded $C^{2,\delta}$ domain.

Approximation Lemma

If $u \in C^{0,\gamma}(\Omega)$ satisfies $\operatorname{div} u = 0$ and $u \cdot n = 0$ on $\partial\Omega$, then there exist

$$u^\varepsilon \in C^\infty(\Omega) \cap C^{1,\delta}(\Omega) \quad \text{with} \quad \operatorname{div} u^\varepsilon = 0 \quad \text{and} \quad u^\varepsilon \cdot n = 0 \text{ on } \partial\Omega$$

such that $u^\varepsilon \to u$ in $C^0(\overline{\Omega})$ as $\varepsilon \to 0^+$ and

$$\sup_{\varepsilon > 0} \|u^\varepsilon\|_{C^{0,\gamma}(\Omega)} \leq C\|u\|_{C^{0,\gamma}(\Omega)}.$$

Representation formula

Let $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ be such that $\operatorname{div} u = 0$ and $u \cdot n| = 0$ on $\partial\Omega$. If p is a weak solution of (P), then

$$p(x) - \frac{1}{|\Omega|}\int_\Omega p(y)\,dy = \int_\Omega \partial_{y_i y_j} G^N_\Omega(x,y)\,(u_i(y) - u_i(x))\,u_j(y)\,dy$$

for all $x \in \Omega$, where $G^N_\Omega = G^N_\Omega(x,y)$ is the Green-Neumann function on Ω.

We argue as in $\mathbb{R}^d$, after proving good decay properties of (derivatives of) G^N_Ω.

Strategy of the proof of Theorem #2 [1/3]

Now $\gamma \in (0, \frac{1}{2}]$ and $\Omega \subset \mathbb{R}^d$ be a simply connected bounded $C^{2,1}$ domain.

By the Approximation Lemma, we may assume $u \in C^\infty \cap C^{1,\delta}$.

Smoothed Theorem #2

If $u \in C^\infty(\Omega) \cap C^{1,\delta}(\Omega)$ satisfies $\operatorname{div} u = 0$ and $u \cdot n| = 0$ on $\partial\Omega$, then there exists a unique zero-average solution $p \in C^{1,\delta}(\Omega)$ of (P) such that

$$\|p\|_{C^{2\gamma}_*(\Omega)} \le C_{\gamma,\Omega} \left(\|u\|^2_{C^{0,\gamma}(\Omega)} + \|p\|_{L^\infty(\Omega)} \right).$$

Here $C^{2\gamma}_* = C^{0,2\gamma}$ for $\gamma < \frac{1}{2}$. No big news for interior, just focus on boundary!

Estimate near the boundary

If $x_0 \in \partial\Omega$, then exists a ball $B_{R_0}(x_0)$ such that, with the notation above,

$$\|p\|_{C^{2\gamma}_*(\Omega \cap B_{R_0}(x_0))} \le C_{\gamma,\Omega,R_0} \left(\|u\|^2_{C^{0,\gamma}(\Omega)} + \|p\|_{L^\infty(\Omega)} \right).$$

Up to paying a suitable $a \in \mathbb{R}$, one reduces to $q = p - \psi$ with $\psi \in C^{1,\min\{\delta,\gamma\}}(\Omega)$ and

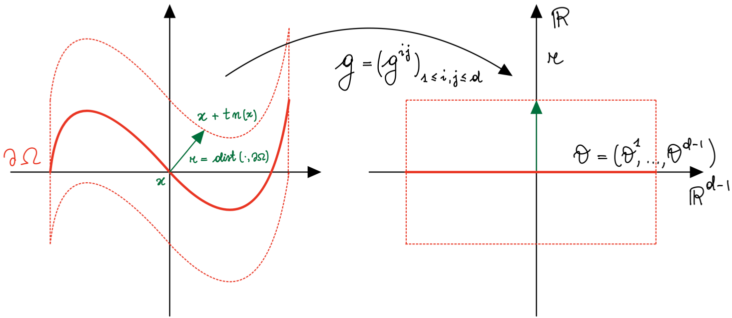
$$\begin{cases} -\Delta q = \operatorname{div}\operatorname{div}(u \otimes u) + a & \text{in } \Omega, \\ \partial_n q = 0 & \text{on } \partial\Omega. \end{cases}$$

Stretching the boundary

We rewrite the problem in a local system o coordinates inside $B_{R_0}(x_0)$ as

$$\begin{cases} -\partial_i \left(w g^{ij}\, \partial_j q \right) = \partial^2_{ij} \left(w\, u^i u^j \right) + w\, a & \text{in } \Omega \cap U \\ \partial_r q(0, \vartheta) = 0 & \text{for } \vartheta \in \Theta. \end{cases}$$

where $w = \sqrt{\det g(r, \vartheta)}$ and ∂_i are the derivatives in the normal coordinates r, ϑ.



Strategy of the proof of Theorem #2 [3/3]

We can reflect for $r < 0$ and then extend to all $\mathbb{R}^d$ with compact support.

The new extended solution $\tilde{q}$ solves a problem of the form

$$-\chi\partial_i\left(\tilde{g}^{ij}\,\partial_j\tilde{q}\right) = \chi\partial^2_{ij}\left(\tilde{w}\,\tilde{u}^i\tilde{u}^j\right) + \mathit{reminder} \quad \text{on } \mathbb{R}^d$$

where χ is a suitable cut-off function.

The reminder term can be neglected, we just need to prove that

$$\|\tilde{q}\|_{C^{2\gamma}_*(\mathbb{R}^d)} \le C\left(\|\tilde{w}\,\tilde{u}\|^2_{C^{0,\gamma}(\mathbb{R}^d)} + \|\tilde{q}\|_{L^\infty(\mathbb{R}^d)}\right).$$

The idea is to study the pseudodifferential operator $D_i = \tilde{g}^{ij}\partial_j$. We may write

$$D = \underbrace{E}_{\text{elliptic}} + \underbrace{L}_{\text{lower order}}$$

and invert the elliptic part via Littlewood-Paley Analysis... painfully technical!

Thank you for your attention!

Slides available via giorgio.stefani.math@gmail.com or giorgiostefani.weebly.com.

[1] L. De Rosa, M. Latocca and GS, On double Hölder regularity of the hydrodynamic pressure in bounded domains, Calc. Var. PDEs 62 (2023), no. 3, Paper No. 85, 31.

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