A distributional approach to fractional Sobolev and BV functions

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NonPUB23 - Nonlocal and Nonlinear Partial Differential Equations

Bologna

June 9, 2023

L. Rizzi ERC Starting Grant 2020 GEOSUB (grant agreement No. 945655)

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No paradoxes without utility

Around 1675 Newton and Leibniz discovered Calculus and nowadays derivative is a basic tool of any mathematician.

Somewhat surprisingly, the first appearance of the concept of a fractional derivative is found in a letter written to De l'Hôpital by Leibniz in 1695!

What is the "half derivative" of x? It's $\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = c\sqrt{x}$ (with $c = \frac{2}{\sqrt{\pi}}$ by Lacroix, 1819).

Leibniz's answer to De L'Hôpital, 30 September 1695:

"Il y a de l'apparence qu'on tirera un jour des consequences bien utiles de ces paradoxes, car il n'y a gueres de paradoxes sans utilité."

"This is an apparent paradox from which, one day, useful consequences will be drawn, since there are no paradoxes without utility."

Leibniz

De L'Hôpital

The fractional derivative: an old story, many definitions

Today there are many fractional derivatives. Three famous examples:

Leibniz-Lacroix (1819): $$\frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Riemann-Liouville (1832-1847): $${}^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha} \, d\tau$$

Caputo (1967): $${}^{C}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha} \, d\tau.$$

Some observations on fractional derivatives:

- they work on functions of just one variable;
- constants can have non-zero fractional derivative;
- they may need differentiable functions!

Question: What about a fractional gradient? Can we just take $(D^\alpha_{e_1}, \ldots, D^\alpha_{e_n})$?

Problem: the 'coordinate approach' does NOT ensure invariance by rotations!

A 'physical' approach: invariance properties

Silhavy proposed that a (physically) 'good' fractional derivative should satisfy:

- invariance with respect to translations and rotations;
- α-homogeneity for some $\alpha \in (0,1)$;
- mild continuity on smooth functions.

For $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, we consider

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(y-x)}{|y-x|^{n+\alpha+1}}\, dy \in \mathbb{R}^n,$$

$$\mathrm{div}^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\, dy \in \mathbb{R},$$

whenever $x \in \mathbb{R}^n$, where $\mu_{n,\alpha} > 0$ is a renormalizing constant.

Characterization Theorem (Silhavy, 2020)

∇^α and div^α are determined (up to mult. const.) by the three requirements above.

A glimpse of the literature

Appearance

1959 Horvath (earliest reference up to knowledge)

1961 Nikol'ski-Sobolev (implicitly mentioned)

Variants, motivated by non-local interactions

1971 Edelen-Laws+Green: thermodynamics & Continuum Mechanics

2011-13-15 Caffarelli-Vazquez+Soria, Biler-Imbert-Karch: porous medium equation

Current research

2015-18 Shieh-Spector: fractional PDE theory (systematic study of ∇^α)

2017-... Spector et al.: optimal embeddings, potential theory

2019-... Comi-S et al: distributional theory for functions and sets

2020-... Silhavy: distributional approach (introducing div^α) & elasticity

2020-... Bellido-Cueto-Mora-Corral: polyconvexity & Continuum Mechanics

2022-... Kreisbeck-Schönberger: quasiconvexity

2023 Braides et al.: homogenizations

Links with fractional Laplacian, Riesz potential/transform and duality

Fractional Laplacian: $-\mathrm{div}^{\beta}\nabla^{\alpha} = (-\Delta)^{\frac{\alpha+\beta}{2}}$, where

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y) - f(x)}{|y-x|^{n+\alpha}}\, dy.$$

Riesz potential: $\nabla^{\alpha} = \nabla I_{1-\alpha}$ and $\mathrm{div}^{\alpha} = \mathrm{div} I_{1-\alpha}$, where

$$I_s f(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}}\, dy, \quad s \in (0, n).$$

Riesz transform: $\nabla^{\alpha} = R\,(-\Delta)^{\frac{\alpha}{2}}$ and $\mathrm{div}^{\alpha} = {}^{t}R\,(-\Delta)^{\frac{\alpha}{2}}$, where

$$Rf(x) = c_n \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(y-x)}{|y-x|^{n+1}}\, dy.$$

Integrability: $f, \varphi \in \mathrm{Lip}_c \implies \nabla^{\alpha} f, \mathrm{div}^{\alpha}\varphi \in L^1 \cap L^{\infty}$.

Duality: the integration-by-parts formula

$$\int_{\mathbb{R}^n} f\, \mathrm{div}^{\alpha}\varphi\, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^{\alpha} f\, dx$$

holds for $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$.

Leibniz's rules (Comi-Stefani, 2019)

Given $f, g \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$, we have

$$\nabla^\alpha(fg) = f\,\nabla^\alpha g + g\,\nabla^\alpha f + \nabla^\alpha_{\mathrm{NL}}(f, g),$$

$$\mathrm{div}^\alpha(f\varphi) = f\mathrm{div}^\alpha\varphi + \varphi \cdot \nabla^\alpha f + \mathrm{div}^\alpha_{\mathrm{NL}}(f, \varphi),$$

where, for $x \in \mathbb{R}^n$, we defined the non-local reminder terms as

$$\nabla^\alpha_{\mathrm{NL}}(f, g)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(g(y) - g(x))(y - x)}{|y - x|^{n+\alpha+1}}\, dy,$$

$$\mathrm{div}^\alpha_{\mathrm{NL}}(f, \varphi)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(\varphi(y) - \varphi(x)) \cdot (y - x)}{|y - x|^{n+\alpha+1}}\, dy.$$

Fundamental Theorem of Fractional Calculus (Silhavy, 2020; Comi-Stefani, 2019)

$$f \in C^\infty_c(\mathbb{R}^n) \implies f(x) = \mu_{n,-\alpha} \int_{\mathbb{R}^n} \frac{\nabla^\alpha f(y) \cdot (y - x)}{|y - x|^{n-\alpha+1}}\, dy \quad \text{for } x \in \mathbb{R}^n.$$

Distributional fractional Sobolev (aka Bessel) functions

For $p \in [1, +\infty]$, we define the distributional fractional Sobolev space

$$S^{\alpha,p}(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n) : \exists\, \nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n) \}$$

endowed with the norm $\|f\|_{S^{\alpha,p}} = \|f\|_{L^p} + \|\nabla^\alpha f\|_{L^p}$.

Here $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ is the weak fractional gradient of $f \in L^p(\mathbb{R}^n)$, i.e.

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n).$$

Identification Theorem (Bruè-Calzi-Comi-S, 2020; Kreisbeck-Schönberger 2022)

If $p \in (1, +\infty)$, then $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$, where

$$L^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : (I - \Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n) \right\}$$

is the Bessel potential space.

Meaning: '$H = W$-type result', since $L^{\alpha,p}(\mathbb{R}^n) = \overline{C^\infty_c(\mathbb{R}^n)}^{\|\cdot\|_{S^{\alpha,p}}}$ for $p \in [1, +\infty)$.

Application: parallel Sobolev theory (PDEs, functionals) for Bessel potential spaces.

Distributional fractional BV functions

Given $p \in [1, +\infty]$, the fractional variation of $f \in L^p(\mathbb{R}^n)$ on an open $\Omega \subset \mathbb{R}^n$ is

$$|D^\alpha f|(\Omega) = \sup\left\{ \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \leq 1 \right\}.$$

We define the distributional $BV^{\alpha,p}(\Omega)$ space

$$BV^{\alpha,p}(\Omega) = \left\{ f \in L^p(\mathbb{R}^n) : |D^\alpha f|(\Omega) < +\infty \right\}$$

endowed with the norm $\|f\|_{BV^{\alpha,p}(\Omega)} = \|f\|_{L^p(\mathbb{R}^n)} + |D^\alpha f|(\Omega)$.

Properties of $BV^{\alpha,p}(\mathbb{R}^n)$ (Comi-S, 2019, Comi-Spector-S, 2022)

Measure: $f \in BV^{\alpha,p}(\Omega) \Longleftrightarrow \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = - \int_\Omega \varphi \cdot dD^\alpha f$ for $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$.

lsc: $|D^\alpha f|(\Omega)$ is l.s.c. with respect to convergence in $L^p(\mathbb{R}^n)$.

Density: $C_c^\infty(\mathbb{R}^n)$ is dense in $BV^{\alpha,p}(\mathbb{R}^n)$ for $p \in \left[1, \frac{n}{n-\alpha}\right)$.

Embedding: $BV^{\alpha,p}(\mathbb{R}^n) \subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ for $p \in \left[1, \frac{n}{n-\alpha}\right)$ and $n \geq 2$.

Compactness: $BV^{\alpha,p}(\mathbb{R}^n) \subset L^p_{\text{loc}}(\mathbb{R}^n)$ is compact for $p \in \left[1, \frac{n}{n-\alpha}\right)$.

Distributional fractional perimeter and fractional reduced boundary

We now focus on the special case $f = \chi_E$ for some measurable set $E \subset \mathbb{R}^n$.

Given any open set $\Omega \subset \mathbb{R}^n$, we let

$$|D^\alpha \chi_E|(\Omega) = \sup\left\{ \int_E \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\}$$

be the distributional fractional (Caccioppoli) perimeter of E inside Ω.

We adopt De Giorgi's idea to define a fractional analogue of the reduced boundary.

The fractional reduced boundary $\mathscr{F}^\alpha E$ (inside Ω) is the set of points

$$x \in \operatorname{supp}(D^\alpha \chi_E) \quad \text{such that} \quad \exists\, \nu_E^\alpha(x) = \lim_{r \to 0^+} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} \in \mathbb{S}^{n-1},$$

where $\nu_E^\alpha(x)$ is the fractional (inner unit) normal at $x \in \Omega \cap \mathscr{F}^\alpha E$.

We thus have the Gauss-Green formula

$$\int_E \operatorname{div}^\alpha \varphi \, dx = - \int_{\Omega \cap \mathscr{F}^\alpha E} \varphi \cdot \nu_E^\alpha \; d|D^\alpha \chi_E|$$

for all $\varphi \in \operatorname{Lip}_c(\Omega; \mathbb{R}^n)$.

Comparison with the $W^{\alpha,p}$ framework [1/2]

For $p \in [1,+\infty)$ and $\alpha \in (0,1)$, we let

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+p\alpha}} \, dx \, dy < +\infty \right\}$$

endowed with the norm $\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^p} + [f]_{W^{\alpha,p}}$.

By definition, $\nabla^\alpha : W^{\alpha,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\nabla^\alpha f\|_{L^1} \le \mu_{n,\alpha} \, [f]_{W^{\alpha,1}}$.

Inclusions #1 (Comi-S, 2019)

- $W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n) \subset BV^{\alpha,1}(\mathbb{R}^n)$ continuously and strictly!
- $0 < \beta < \alpha < 1 \implies BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$

Inclusions #2 (Comi-S, 2019; Bruè-Calzi-Comi-S, 2022)

- $p \in (1,+\infty) \implies S^{\alpha+\varepsilon,p}(\mathbb{R}^n) \subset W^{\alpha,p}(\mathbb{R}^n) \subset S^{\alpha-\varepsilon,p}(\mathbb{R}^n)$
- $p \in [1,2) \implies W^{\alpha,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$
- $W^{\alpha,2}(\mathbb{R}^n) = S^{\alpha,2}(\mathbb{R}^n)$
- $p \in (2,+\infty]$ and $0 < \alpha < \beta < 1 \implies W^{\beta,1}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$

Comparison with the $W^{\alpha,p}$ framework [2/2]

The fractional perimeter in an open set $\Omega \subset \mathbb{R}^n$ of a measurable set $E \subset \mathbb{R}^n$ is

$$P_\alpha(E;\Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}}\, dx\, dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}}\, dx\, dy.$$

If $\Omega = \mathbb{R}^n$, then $P_\alpha(E;\mathbb{R}^n) = P_\alpha(E) = [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)}$.

Representation formulas (Comi-S, 2019)

- $P_\alpha(E;\Omega) < +\infty \implies |D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E;\Omega)$ with $D^\alpha \chi_E = \nabla^\alpha \chi_E \mathscr{L}^n$
- $\chi_E \in BV(\mathbb{R}^n) \implies \nabla^\alpha \chi_E(x) = \dfrac{\mu_{n,\alpha}}{n+\alpha-1} \displaystyle\int_{\mathbb{R}^n} \frac{\nu_E(y)}{|y-x|^{n+\alpha-1}}\, d|D\chi_E|(y)$
- $|D^\alpha \chi_E| \ll \mathscr{L}^n$ and $|D^\alpha \chi_E|(\Omega) > 0 \implies \mathscr{L}^n(\Omega \cap \mathscr{F}^\alpha E) > 0$!

Careful: $P_\alpha(E;\Omega) < +\infty \implies \mathscr{F}^\alpha E$ is diffuse (recall that $W^{\alpha,1} \subset S^{\alpha,1}$).

Example: $E = (a,b) \subset \mathbb{R} \Rightarrow \mathscr{F}^\alpha E = \mathbb{R} \setminus \left\{\frac{a+b}{2}\right\}$!

Two examples: the ball and the halfspace

Recall that ∇^α is invariant by translations and rotations in $\mathbb{R}^n$.

Ball. For $x \in \mathbb{R}^n$ with $|x| \neq 0, 1$, we have

$$\nabla^\alpha \chi_{B_1}(x) = -\frac{\mu_{n,\alpha}}{n+\alpha-1}\, g_{n,\alpha}(|x|)\, \frac{x}{|x|},$$

where

$$g_{n,\alpha}(t) = \int_{\partial B_1} \frac{y_1}{|t\mathrm{e}_1 - y|^{n+\alpha-1}}\, d\mathscr{H}^{n-1}(y) > 0, \quad \text{for any } t \geq 0.$$

As a consequence, $\nu^\alpha_{B_1}(x) = -\dfrac{x}{|x|}$ for any $x \neq 0$ and $\mathscr{F}^\alpha B_1 = \mathbb{R}^n \setminus \{0\}$.

Halfspace. Letting $H^+_\nu = \{x \in \mathbb{R}^n : x \cdot \nu \geq 0\}$ for $\nu \in \mathbb{S}^{n-1}$, if $x \cdot \nu \neq 0$ then

$$\nabla^\alpha \chi_{H^+_\nu}(x) = \frac{2^{\alpha-1}\Gamma\left(\frac{\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1-\alpha}{2}\right)}\, \frac{1}{|x \cdot \nu|^\alpha}\, \nu.$$

As a consequence, $\mathscr{F}^\alpha H^+_\nu = \mathbb{R}^n$ and $\nu^\alpha_{H^+_\nu} \equiv \nu$.

Density estimates

Now we go back to sets with locally finite distributional fractional perimeter.

The fractional reduced boundary allows for local upper density estimates.

Density estimates (Comi-S, 2019)

If $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathscr{F}^\alpha E$, then there exists $r_x > 0$ such that

$$|D^\alpha \chi_E|(B_r(x)) \le A_{n,\alpha} r^{n-\alpha} \quad \text{and} \quad |D^\alpha \chi_{E \cap B_r(x)}|(\mathbb{R}^n) \le B_{n,\alpha} r^{n-\alpha}$$

for all $r \in (0, r_x)$, where $A_{n,\alpha}, B_{n,\alpha} > 0$ are universal constants.

Integration-by-parts on balls (Comi-S, 2019)

If $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathscr{F}^\alpha E$, then for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$ and a.e. $r > 0$

$$\int_{E \cap B_r(x)} \mathrm{div}^\alpha \varphi \, dy + \int_E \varphi \cdot \nabla^\alpha \chi_{B_r(x)} \, dy + \int_E \mathrm{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \varphi) \, dy = -\int_{B_r(x)} \varphi \cdot D^\alpha \chi_E.$$

By a standard covering argument, we thus get that

$$|D^\alpha \chi_E| \le C_{n,\alpha} \mathscr{H}^{n-\alpha} \llcorner \mathscr{F}^\alpha E \quad \text{so that} \quad \dim_{\mathscr{H}}(\mathscr{F}^\alpha E) \ge n - \alpha.$$

Recall that $\dim_{\mathscr{H}}(\mathscr{F}^\alpha E) = n$ in the $W^{\alpha,1}$ regime (in particular, for BV sets)!

Fractional De Giorgi's Blow-up Theorem

The blow-ups at $x \in \mathbb{R}^n$ of a set $E \subset \mathbb{R}^n$ are the family

$$\mathrm{Tan}(E,x) = \left\{ \text{limit points of } \left(\tfrac{E-x}{r}\right)_{r>0} \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+ \right\}.$$

We can prove a fractional analogue of De Giorgi's Blow-up Theorem.

Fractional Blow-up Theorem (Comi-S, 2019)

If $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathscr{F}^\alpha E$, then:

- existence: $\mathrm{Tan}(E,x) \neq \emptyset$;
- rigidity: $F \in \mathrm{Tan}(E,x) \implies \nu^\alpha_F(y) = \nu^\alpha_E(x)$ for $|D^\alpha \chi_F|$-a.e. $y \in \mathscr{F}^\alpha F$.

Open Problem: How to characterize blow-ups?

Bad News Theorem (Comi-S, 2019 and 2022)

The coarea formula and the local chain rule do NOT hold!

$$|D^\alpha f| \neq \int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|\, dt \qquad \text{and} \qquad |D^\alpha \Phi(f)| \neq |\Phi'|\,|D^\alpha f|$$

Further properties of blow-ups

Let $\chi_E \in BV^{\alpha,\infty}_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathscr{F}^\alpha E$ and assume that $\nu^\alpha_E(x) = \vec{e}_n$.

Properties of blow-ups (Comi-S, 2023)

If $F \in \tan(E,x)$, then $F = \mathbb{R}^{n-1} \times M$ with $M \subset \mathbb{R}$ such that

- $\chi_M \in BV^{\alpha,\infty}_{\text{loc}}(\mathbb{R})$ with $\partial^\alpha \chi_M \geq 0$;
- $|M|, |M^c| \in \{0, +\infty\}$;
- $|M| = +\infty \implies \operatorname{ess\,sup} M = +\infty$;
- $M \neq \emptyset, \mathbb{R}$ is such that $P(M) < +\infty \implies M = (m, +\infty)$ for some $m \in \mathbb{R}$.

Consequence: halfspaces and $\mathbb{R}^n$ are the ONLY blow-up cones!

Splitting Theorem (Comi-S, 2023)

Assume $f \in BV^{\alpha,\infty}_{\text{loc}}(\mathbb{R}^n)$.

- $D^\alpha_i f = 0 \iff D_i f = 0$ whatever $i \in \{1, \dots, n\}$
- $D_1 f = 0 \implies \exists g \in BV^{\alpha,\infty}_{\text{loc}}(\mathbb{R}^{n-1})$ such that $f((x_1, x')) = g(x')$ a.e. and

$$(D^\alpha_{\mathbb{R}^n})_i f = \mathscr{L}^1 \otimes (D^\alpha_{\mathbb{R}^{n-1}})_i g \quad \text{for } i = 2, \dots, n.$$

Non-local boundaries [1/2]

$\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \implies D^\alpha\chi_E = D^\alpha_{\mathrm{ac}}\chi_E + D^\alpha_{\mathrm{s}}\chi_E,\ D^\alpha_{\mathrm{ac}}\chi_E \ll \mathscr{L}^n,\ D^\alpha_{\mathrm{s}}\chi_E \perp \mathscr{L}^n$

Locality of singular part (Schönberger, 2023; Comi-S, 2023)

$$D^\alpha_{\mathrm{s}}\chi_{E\cap F} = \chi_{F^1} D^\alpha_{\mathrm{s}}\chi_E \text{ whenever } P_\alpha(F) < +\infty$$

Notation: $F^t = \left\{x \in \mathbb{R}^n : \exists \lim_{r\to 0^+} \frac{|F \cap B_r(x)|}{|B_r(x)|} = t\right\}$ whenever $t \in [0,1]$.

Support of singular part (Comi-S, 2023)

$D^\alpha_{\mathrm{s}}\chi_E(F^1) = 0$ whenever $\chi_F \in W^{\alpha,1}(\mathbb{R}^n)$ with either $|E \cap F| = 0$ or $|E^c \cap F| = 0$

Notation: $\partial^- E = \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < |B_r(x)| \text{ for all } r > 0\}$.

Corollary: $\mathrm{supp}(|D^\alpha_{\mathrm{s}}\chi_E|) \subset \partial^- E$ and so $|D^\alpha_{\mathrm{s}}\chi_E| \le C_{n,\alpha}\,\mathscr{H}^{n-\alpha} \llcorner (\mathscr{F}^\alpha E \cap \partial^- E)$

Analogies with classical results (De Giorgi's Theory & Lombardini, 2019)

- $\chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^n) \implies \mathrm{supp}(|D\chi_E|) = \partial^- E$
- $\chi_E \in W^{\alpha,1}_{\mathrm{loc}}(\mathbb{R}^n) \implies \partial^- E = \left\{x \in \mathbb{R}^n : [\chi_E]_{W^{\alpha,1}(B_r(x))} > 0 \text{ for all } r > 0\right\}$

Caution: there exists $E \subset \mathbb{R}^n$ such that $P(E) < +\infty$ and $\mathscr{L}^n(\partial^- E) = +\infty$!

Non-local boundaries [2/2]

Exercise (measure-theoretic interior and exterior)

If $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is measurable, then

$$\mathrm{Tan}(E,x) = \{\mathbb{R}^n\} \iff x \in E^1 \quad \text{and} \quad \mathrm{Tan}(E,x) = \{\emptyset\} \iff x \in E^0$$

Notation: $\partial^* E = \mathbb{R}^n \setminus (E^0 \cup E^1)$ is the measure-theoretic boundary.

Definition (Effective fractional reduced boundary)

We define $\mathscr{F}^\alpha_{\mathrm{e}} E = \mathscr{F}^\alpha E \cap \partial^* E$ whenever $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$.

Properties of effective fractional reduced boundary (Comi-S, 2023)

- $\chi_E \in BV^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathscr{F}^\alpha_{\mathrm{e}} E$, $\mathrm{Tan}(E,x) = H^+_\nu$ for $\nu \in \mathbb{S}^{n-1} \implies \nu^\alpha_E(x) = \nu$
- $\chi_E \in W^{\alpha,1}_{\mathrm{loc}}(\mathbb{R}^n) \implies \mathscr{H}^{n-\alpha}(\mathscr{F}^\alpha_{\mathrm{e}} E) = 0$
- $\chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^n) \implies \mathscr{F} E \subset \mathscr{F}^\alpha_{\mathrm{e}} E$, $\mathscr{H}^{n-1}(\mathscr{F}^\alpha_{\mathrm{e}} E \setminus \mathscr{F} E) = 0$, $\nu^\alpha_E = \nu_E$ on $\mathscr{F} E$

Finer analysis of $BV^{\alpha,p}$ functions

Absolute continuity properties of the fractional variation (Comi-Spector-S, 2022)

$$f \in BV^{\alpha,p}(\mathbb{R}^n) \implies \begin{cases} |D^\alpha f| \ll \mathscr{H}^{n-1} & \text{for } p \in \left[1, \frac{n}{1-\alpha}\right) \\ |D^\alpha f| \ll \mathscr{H}^{n-\alpha-\frac{n}{p}} & \text{for } p \in \left(\frac{n}{1-\alpha}, +\infty\right] \end{cases}$$

The precise representative of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is $f^\star(x) = \lim_{r \to 0^+} \fint_{B_r(x)} f(y)\,dy$.

Precise representative (Comi-Spector-S, 2022)

$$f \in BV^{\alpha,p}(\mathbb{R}^n) \implies \lim_{r \to 0^+} \fint_{B_r(x)} |f(y) - f^\star(x)|^q\,dy = 0 \quad \mathscr{H}^{n-\alpha+\varepsilon}\text{-a.e. } x \in \mathbb{R}^n$$

for any $q \in [1, \bar{q}_\varepsilon]$, where $\bar{q}_\varepsilon \in \left[1, \frac{n}{n-\alpha}\right)$ is such that $\lim_{\varepsilon \to 0^+} \bar{q}_\varepsilon = \frac{n}{n-\alpha}$

Leibniz's rules for $BV^{\alpha,p}$ with Besov (Comi-S, 2022)

If $f \in BV^{\alpha,p}(\mathbb{R}^n)$ and $g \in$ Besov, then $fg \in BV^{\alpha,p}(\mathbb{R}^n)$ with

$$D^\alpha(fg) = g^\star D^\alpha f + f\,\nabla^\alpha g\,\mathscr{L}^n + \nabla^\alpha_{\mathrm{NL}}(f,g)\,\mathscr{L}^n \quad \text{in } \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n).$$

Fractional divergence-measure fields

Given $p \in [1, +\infty]$, $F \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ has fractional α-divergence-measure, and we write $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, if there exists $divF \in \mathscr{M}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \, dx = -\int_{\mathbb{R}^n} \xi \, ddiv^\alpha F \qquad \text{for all } \xi \in C_c^\infty(\mathbb{R}^n).$$

Absolute continuity properties of the fractional divergence-measure (Comi-S, 2023)

$$F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n) \implies \begin{cases} \text{nothing} & \text{for } p \in \left[1, \frac{n}{n-\alpha}\right) \\ div^\alpha F \equiv 0 \text{ on Borel sets with } \sigma\text{-finite} & \\ \mathscr{H}^{n - \frac{p}{p-1+(1-\alpha)\frac{p}{n}}} \text{ measure} & \text{for } p \in \left[\frac{n}{n-\alpha}, \frac{n}{1-\alpha}\right) \\ |div^\alpha F| \ll \mathscr{H}^{n-\alpha-\frac{n}{p}} & \text{for } p \in \left[\frac{n}{1-\alpha}, +\infty\right] \end{cases}$$

Remark: the case $\alpha = 1$ is due to [Silhavy, 2005].

Leibniz's rules for $\mathcal{DM}^{\alpha,p}$ with Besov (Comi-S, 2023)

If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ and $g \in$ Besov, then $gF \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ with

$$div^\alpha(gF) = g^\star div^\alpha F + F \cdot \nabla^\alpha g \, \mathscr{L}^n + div^\alpha_{\mathsf{NL}}(g, F) \, \mathscr{L}^n \quad \text{in } \mathscr{M}(\mathbb{R}^n).$$

Asymptotics

Now important: $\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \sim \frac{1-\alpha}{|B_1|}$ as $\alpha \to 1^-$.

Asymptotics for $\alpha \to 1^-$ (Comi-S, 2019) → Bourgain-Brezis-Mironescu

- $f \in W^{1,p}(\mathbb{R}^n) \implies \nabla^\alpha f \to \nabla f$ in L^p whenever $p \in [1, +\infty)$.
- $f \in BV(\mathbb{R}^n) \implies D^\alpha f \rightharpoonup Df$, $|D^\alpha f| \rightharpoonup |Df|$, $|D^\alpha f|(\mathbb{R}^n) \to |Df|(\mathbb{R}^n)$.

Asymptotics for $\alpha \to 0^+$ (Bruè-Calzi-Comi-S, 2020) → Maz'ya-Shaposhnikova

- $f \in \bigcup_{\alpha\in(0,1)} W^{\alpha,p}(\mathbb{R}^n) \implies \nabla^\alpha f \to Rf$ in L^p whenever $p \in (1, +\infty)$.
- $f \in H^1(\mathbb{R}^n) \cap \bigcup_{\alpha\in(0,1)} W^{\alpha,1}(\mathbb{R}^n) \implies \nabla^\alpha f \to Rf$ in L^1 and in H^1.
- $f \in \bigcup_{\alpha\in(0,1)} W^{\alpha,1}(\mathbb{R}^n) \implies \alpha \int_{\mathbb{R}^n} |\nabla^\alpha f(x)|\, dx \to c_n \left| \int_{\mathbb{R}^n} f(x)\, dx \right|$.

Notation: The Hardy space is $H^1 = \left\{ f \in L^1(\mathbb{R}^n) : Rf \in L^1(\mathbb{R}^n) \right\}$ (with $R = \nabla^0$).

Update: distributional BBM formula obtained by [Brezis-Mironescu, April 2023]!

Fractional interpolation inequalities

We prove $\nabla^\alpha \to R$ strongly as $\alpha \to 0^+$ via fractional interpolation inequalities.

Interpolation #1 (Bruè-Calzi-Comi-S, 2020)

Let $\alpha \in (0,1]$. There exists $c_{n,\alpha} > 0$ such that

$$|D^\beta f|(\mathbb{R}^n) \le c_{n,\alpha} \|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} |D^\alpha f|(\mathbb{R}^n)^{\frac{\beta}{\alpha}}$$

for all $\beta \in [0,\alpha]$ and all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$.

Interpolation #2 (Bruè-Calzi-Comi-S, 2020)

Let $p \in (1,+\infty)$. There exists $c_{n,p} > 0$ such that

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,p} \|\nabla^\gamma f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$. There exists $c_n > 0$ such that

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)} \le c_n \|\nabla^\gamma f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in HS^{\alpha,1}(\mathbb{R}^n)$.

$HS^{\alpha,1}(\mathbb{R}^n) = \{f \in H^1(\mathbb{R}^n) : \nabla^\alpha f \in H^1(\mathbb{R}^n;\mathbb{R}^n)\}$ is the Hardy-Sobolev space.

Definition (Γ-convergence after De Giorgi)

We say that $F_k \colon (X, \mathsf{d}) \to \overline{\mathbb{R}}$ Γ-converge to $F \colon (X, \mathsf{d}) \to \overline{\mathbb{R}}$ if

- $x_k \to x \implies F(x) \le \liminf_k F(x_k)$
- $\forall x \in X \; \exists x_k \to x$ such that $F(x) \ge \limsup_k F_k(x_k)$

Γ-conv. as $\alpha \to 1^-$ (Comi-S, 2022) $\to$ Ambrosio-De Philippis-Martinazzi & Ponce

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary or $\Omega = \mathbb{R}^n$.

- $f \in BV(\Omega) \implies \Gamma(L^1)\text{ - }\lim_{\alpha \to 1^-} |D^\alpha f|(\Omega) = |Df|(\Omega)$
- $E \subset \mathbb{R}^n$ measurable $\implies \Gamma(L^1_{\mathrm{loc}})\text{ - }\lim_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) = P(E; \Omega)$

Why is Γ-convergence interesting? Minimizers of F_k converge to minimizers of F!

Possible application: regularity of distributional non-local minimal surfaces.

Update: similar Γ-convergence results obtained by [Brezis-Mironescu, April 2023]!

Remark: asymptotics and Γ-convergence can be generalized to $\alpha \to \alpha_0^- \in (0, 1]$.

Open problems and research directions

About sets and perimeter

- ▷ Is there a set $E \subset \mathbb{R}^n$ such that $\chi_E \in BV^{\alpha,1}(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n)$?
- ▷ Is there a set with $\mathscr{L}^n$-negligible fractional reduced boundary?
- ▷ What are 1D profiles of blow-ups?
- ▷ Are balls isoperimetric sets for the fractional variation?
- ▷ What about minimal surfaces for the fractional variation (existence, regularity)?

About functions and variation

- ▷ Do $BV^{\alpha,p}$ functions satisfy some good local properties?
  - does the precise representative exist $\mathscr{H}^{n-\alpha}$-a.e. in $\mathbb{R}^n$?
  - Are there approximate limits? What are fractional jumps?
- ▷ Do functions in $BV^{\alpha,\infty}(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n)$ satisfy a Leibniz's rule?
- ▷ Is there a Leibniz's rule for $F \in \mathcal{DM}^{\alpha,\infty}(\mathbb{R}^n)$ and $\chi_E \in BV^{\alpha,1}(\mathbb{R}^n)$?
- ▷ Can $BV^{\alpha,p}$ functions be defined on a general open set $\Omega \subset \mathbb{R}^n$?

Thank you for your attention!

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