

A distributional approach to fractional Sobolev and BV functions I

Giorgio Stefani

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Louvain-la-Neuve



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Project and collaborators

- [1] [G. E. Comi](#) and GS, A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up, *JFA* (2019)
- [2] [G. E. Comi](#) and GS, A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics I, *RMC* (2022)
- [3] [E. Bruè](#), [M. Calzi](#), [G. E. Comi](#) and GS, A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics II, *CR Math* (2020)
- [4] [G. E. Comi](#), [D. Spector](#) and GS, The fractional variation and the precise representative of $BV^{\alpha,p}$ functions, *FCAA* (2022)
- [5] [G. E. Comi](#) and G. Stefani, [Leibniz rules](#) and Gauss-Green formulas in distributional fractional spaces, *JMAA* (2022)
- [6] [G. E. Comi](#) and GS, Failure of the local chain rule for the fractional variation, *PM* (2023)
- [7] [G. E. Comi](#) and GS, Fractional divergence-measure fields, Leibniz rule and Gauss-Green formula, *arXiv* (2023)
- [8] [G. E. Comi](#) and GS, On sets with finite distributional fractional perimeter, *arXiv* (2023)

No paradoxes without utility

Around 1675 Newton and Leibniz discovered Calculus and nowadays derivative is a basic tool of any mathematician.

Somewhat surprisingly, the first appearance of the concept of a fractional derivative is found in a letter written to De l'Hôpital by Leibniz in 1695!

What is the "half derivative" of x ? It's $\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = c\sqrt{x}$ (with $c = \frac{2}{\sqrt{\pi}}$ by Lacroix, 1819).

Leibniz's answer to De L'Hôpital, 30 September 1695:

"Il y a de l'apparence qu'on tirera un jour des consequences bien utiles de ces paradoxes, car il n'y a gueres de paradoxes sans utilité."

"This is an apparent paradox from which, one day, useful consequences will be drawn, since **there are no paradoxes without utility.**"



The fractional derivative: an old story, many definitions

Today there are many **fractional derivatives**. Three famous examples:

$$\text{Leibniz-Lacroix (1819):} \quad \frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

$$\text{Riemann-Liouville (1832-1847):} \quad {}^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau$$

$$\text{Caputo (1967):} \quad {}^C D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau.$$

Some observations on fractional derivatives:

- they work on functions of just **one variable**;
- **constants** can have non-zero fractional derivative;
- they may need **differentiable** functions!

Question: What about a **fractional gradient**? Can we just take $(D_{e_1}^\alpha, \dots, D_{e_n}^\alpha)$?

Problem: the 'coordinate approach' does **NOT** ensure invariance by rotations!

A 'physical' approach: invariance properties

Silhavy proposed that a (physically) 'good' fractional derivative should satisfy:

- **invariance** with respect to translations and rotations;
- **α -homogeneity** for some $\alpha \in (0, 1)$;
- mild **continuity** on smooth functions.

For $f \in \text{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \text{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$, we consider

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(y - x)}{|y - x|^{n+\alpha+1}} dy \in \mathbb{R}^n,$$

$$\text{div}^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y) - \varphi(x)) \cdot (y - x)}{|y - x|^{n+\alpha+1}} dy \in \mathbb{R},$$

whenever $x \in \mathbb{R}^n$, where $\mu_{n,\alpha} > 0$ is a renormalizing constant.

Theorem (Silhavy, 2020)

∇^α and div^α are determined (up to mult. const.) by the three requirements above.

A glimpse of the literature

Appearance

1959 Horvath (earliest reference up to knowledge)

1961 Nikol'ski-Sobolev (implicitly mentioned)

Variants, motivated by non-local interactions

1971 Edelen-Laws+Green: thermodynamics & Continuum Mechanics

2011-13-15 Caffarelli-Vazquez+Soria, Biler-Imbert-Karch: porous medium equation

Current research

2015-18 Shieh-Spector: fractional PDE theory (systematic study of ∇^α)

2017-... Spector et al.: optimal embeddings, potential theory

2019-... Comi-S et al: distributional theory for functions and sets

2020-... Silhavy: distributional approach (introducing $\operatorname{div}^\alpha$) & elasticity

2020-... Bellido-Cueto-Mora-Corral: polyconvexity & Continuum Mechanics

2022-... Kreisbeck-Schönberger: quasiconvexity

2023 Braides et al.: homogenizations

Links with fractional Laplacian, Riesz potential/transform and duality

Fractional Laplacian: $-\operatorname{div}^\beta \nabla^\alpha = (-\Delta)^{\frac{\alpha+\beta}{2}}$, where

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y) - f(x)}{|y-x|^{n+\alpha}} dy.$$

Riesz potential: $\nabla^\alpha = \nabla I_{1-\alpha}$ and $\operatorname{div}^\alpha = \operatorname{div} I_{1-\alpha}$, where

$$I_s f(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} dy, \quad s \in (0, n).$$

Riesz transform: $\nabla^\alpha = R(-\Delta)^{\frac{\alpha}{2}}$ and $\operatorname{div}^\alpha = {}^t R(-\Delta)^{\frac{\alpha}{2}}$, where

$$Rf(x) = c_n \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(y-x)}{|y-x|^{n+1}} dy.$$

Integrability: $f, \varphi \in \operatorname{Lip}_c \implies \nabla^\alpha f, \operatorname{div}^\alpha \varphi \in L^1 \cap L^\infty$.

Duality: the integration-by-parts formula

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi dx = - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f dx$$

holds for $f \in \operatorname{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$.

Leibniz's rules for ∇^α and $\operatorname{div}^\alpha$

If $f, g \in \operatorname{Lip}_c(\mathbb{R}^n)$, then

$$\nabla^\alpha(fg) = f \nabla^\alpha g + g \nabla^\alpha f + \nabla_{\text{NL}}^\alpha(f, g),$$

where

$$\nabla_{\text{NL}}^\alpha(f, g)(x) = \mu_{n, \alpha} \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(g(y) - g(x))(y - x)}{|y - x|^{n+\alpha+1}} dy, \quad x \in \mathbb{R}^n.$$

Similarly, if $f \in \operatorname{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$, then

$$\operatorname{div}^\alpha(f\varphi) = f \operatorname{div}^\alpha \varphi + \varphi \cdot \nabla^\alpha f + \operatorname{div}_{\text{NL}}^\alpha(f, \varphi),$$

where

$$\operatorname{div}_{\text{NL}}^\alpha(f, \varphi)(x) = \mu_{n, \alpha} \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(\varphi(y) - \varphi(x)) \cdot (y - x)}{|y - x|^{n+\alpha+1}} dy, \quad x \in \mathbb{R}^n.$$

Approximation scheme

- ∇^α & $\operatorname{div}^\alpha$ linear \implies use **convolution** \implies smoothing
- Leibniz's rules (check NL!) \implies use **cut-off functions** \implies compact support

A fractional version of the Fundamental Theorem of Calculus

Theorem (Silhavy, 2020; Comi-Stefani, 2019)

If $f \in C_c^\infty(\mathbb{R}^n)$, then

$$f(x) = \mu_{n,-\alpha} \int_{\mathbb{R}^n} \frac{\nabla^\alpha f(y) \cdot (y-x)}{|y-x|^{n-\alpha+1}} dy$$

for any $x, y \in \mathbb{R}^n$ (i.e., $f = \operatorname{div}^{-\alpha} \nabla^\alpha f$).

Good news:

- L^p -control on translations \implies compactness for bounded sequences
- $\operatorname{RHS} \lesssim I_\alpha |\nabla^\alpha f| \implies$ Sobolev embedding for $p \in (1, +\infty)$

Bad news:

- RHS on whole space \implies NO local Poincaré inequality
- RHS delicate for $p = 1 \implies$ finer analysis in the fractional BV setting

Distributional fractional Sobolev (aka Bessel) functions

For $p \in [1, +\infty]$, we define the **distributional fractional Sobolev space**

$$S^{\alpha,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \exists \nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n)\}$$

endowed with the norm $\|f\|_{S^{\alpha,p}} = \|f\|_{L^p} + \|\nabla^\alpha f\|_{L^p}$.

Here $\nabla^\alpha f \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ is the **weak fractional gradient** of $f \in L^p(\mathbb{R}^n)$, i.e.

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n).$$

Theorem (Bruè-Calzi-Comi-S, 2020; Kreisbeck-Schönberger 2022)

If $p \in (1, +\infty)$, then $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$, where

$$L^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : (I - \Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n) \right\}$$

is the **Bessel potential space**.

Meaning: 'H = W-type result', since $L^{\alpha,p}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{S^{\alpha,p}}}$ for $p \in [1, +\infty)$.

Application: parallel Sobolev theory (PDEs, functionals) for Bessel potential spaces.

Distributional fractional BV functions

Given $p \in [1, +\infty]$, the **fractional variation** of $f \in L^p(\mathbb{R}^n)$ on an open $\Omega \subset \mathbb{R}^n$ is

$$|D^\alpha f|(\Omega) = \sup \left\{ \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \|\varphi\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \leq 1 \right\}.$$

We define the **distributional $BV^{\alpha,p}(\Omega)$ space**

$$BV^{\alpha,p}(\Omega) = \left\{ f \in L^p(\mathbb{R}^n) : |D^\alpha f|(\Omega) < +\infty \right\}$$

endowed with the norm $\|f\|_{BV^{\alpha,p}(\Omega)} = \|f\|_{L^p(\mathbb{R}^n)} + |D^\alpha f|(\Omega)$.

Theorem (Comi-S, 2019, Comi-Spector-S, 2022)

Measure: $f \in BV^{\alpha,p}(\Omega) \iff \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = - \int_{\Omega} \varphi \cdot dD^\alpha f$ for $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$.

lsc: $|D^\alpha f|(\Omega)$ is l.s.c. with respect to convergence in $L^p(\mathbb{R}^n)$.

Density: $C_c^\infty(\mathbb{R}^n)$ is dense in $BV^{\alpha,p}(\mathbb{R}^n)$ for $p \in \left[1, \frac{n}{n-\alpha}\right)$.

Embedding: $BV^{\alpha,p}(\mathbb{R}^n) \subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ for $p \in \left[1, \frac{n}{n-\alpha}\right)$ and $n \geq 2$.

Compactness: $BV^{\alpha,p}(\mathbb{R}^n) \subset L^p_{\text{loc}}(\mathbb{R}^n)$ is compact for $p \in \left[1, \frac{n}{n-\alpha}\right)$.

Distributional fractional perimeter and fractional reduced boundary

We now focus on the special case $f = \chi_E$ for some measurable set $E \subset \mathbb{R}^n$.

Given any open set $\Omega \subset \mathbb{R}^n$, we let

$$|D^\alpha \chi_E|(\Omega) = \sup \left\{ \int_E \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \|\varphi\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 1 \right\}$$

be the **distributional fractional (Caccioppoli) perimeter** of E inside Ω .

We adopt De Giorgi's idea to define a fractional analogue of the reduced boundary.

The **fractional reduced boundary** $\mathcal{F}^\alpha E$ (inside Ω) is the set of points

$$x \in \operatorname{supp}(D^\alpha \chi_E) \quad \text{such that} \quad \exists \nu_E^\alpha(x) = \lim_{r \rightarrow 0^+} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} \in \mathbb{S}^{n-1},$$

where $\nu_E^\alpha(x)$ is the **fractional (inner unit) normal** at $x \in \Omega \cap \mathcal{F}^\alpha E$.

We thus have the **Gauss-Green formula**

$$\int_E \operatorname{div}^\alpha \varphi \, dx = - \int_{\Omega \cap \mathcal{F}^\alpha E} \varphi \cdot \nu_E^\alpha \, d|D^\alpha \chi_E|$$

for all $\varphi \in \operatorname{Lip}_c(\Omega; \mathbb{R}^n)$.

Comparison with the $W^{\alpha,p}$ framework [1/2]

For $p \in [1, +\infty)$ and $\alpha \in (0, 1)$, we let

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : [f]_{W^{\alpha,p}(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\alpha}} dx dy < +\infty \right\}$$

endowed with the norm $\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^p} + [f]_{W^{\alpha,p}}$.

By definition, $\nabla^\alpha : W^{\alpha,1}(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\nabla^\alpha f\|_{L^1} \leq \mu_{n,\alpha} [f]_{W^{\alpha,1}}$.

Theorem (Comi-S, 2019)

- $W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n) \subset BV^{\alpha,1}(\mathbb{R}^n)$ continuously and **strictly!**
- $0 < \beta < \alpha < 1 \implies BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$

Theorem (Comi-S, 2019; Bruè-Calzi-Comi-S, 2022)

- $p \in (1, +\infty) \implies S^{\alpha+\varepsilon,p}(\mathbb{R}^n) \subset W^{\alpha,p}(\mathbb{R}^n) \subset S^{\alpha-\varepsilon,p}(\mathbb{R}^n)$
- $p \in [1, 2) \implies W^{\alpha,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$
- $W^{\alpha,2}(\mathbb{R}^n) = S^{\alpha,2}(\mathbb{R}^n)$
- $p \in (2, +\infty]$ and $0 < \alpha < \beta < 1 \implies W^{\beta,1}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$

Comparison with the $W^{\alpha,p}$ framework [2/2]

The fractional perimeter in an open set $\Omega \subset \mathbb{R}^n$ of a measurable set $E \subset \mathbb{R}^n$ is

$$P_\alpha(E; \Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+\alpha}} dx dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+\alpha}} dx dy.$$

If $\Omega = \mathbb{R}^n$, then $P_\alpha(E; \mathbb{R}^n) = P_\alpha(E) = [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)}$.

Theorem (Comi-S, 2019)

- $P_\alpha(E; \Omega) < +\infty \implies |D^\alpha \chi_E|(\Omega) \leq \mu_{n,\alpha} P_\alpha(E; \Omega)$ with $D^\alpha \chi_E = \nabla^\alpha \chi_E \mathcal{L}^n$
- $\chi_E \in BV(\mathbb{R}^n) \implies \nabla^\alpha \chi_E(x) = \frac{\mu_{n,\alpha}}{n + \alpha - 1} \int_{\mathbb{R}^n} \frac{\nu_E(y)}{|y - x|^{n+\alpha-1}} d|D\chi_E|(y)$
- $|D^\alpha \chi_E| \ll \mathcal{L}^n$ and $|D^\alpha \chi_E|(\Omega) > 0 \implies \mathcal{L}^n(\Omega \cap \mathcal{F}^\alpha E) > 0!$

Careful: $P_\alpha(E; \Omega) < +\infty \implies \mathcal{F}^\alpha E$ is **diffuse** (recall that $W^{\alpha,1} \subset S^{\alpha,1}$).

Example: $E = (a,b) \subset \mathbb{R} \implies \mathcal{F}^\alpha E = \mathbb{R} \setminus \{\frac{a+b}{2}\}!$

Two examples: the ball and the halfspace

Recall that ∇^α is invariant by translations and rotations in \mathbb{R}^n .

Ball. For $x \in \mathbb{R}^n$ with $|x| \neq 0, 1$, we have

$$\nabla^\alpha \chi_{B_1}(x) = -\frac{\mu_{n,\alpha}}{n + \alpha - 1} g_{n,\alpha}(|x|) \frac{x}{|x|},$$

where

$$g_{n,\alpha}(t) = \int_{\partial B_1} \frac{y_1}{|te_1 - y|^{n+\alpha-1}} d\mathcal{H}^{n-1}(y) > 0, \quad \text{for any } t \geq 0.$$

As a consequence, $\nu_{B_1}^\alpha(x) = -\frac{x}{|x|}$ for any $x \neq 0$ and $\mathcal{F}^\alpha B_1 = \mathbb{R}^n \setminus \{0\}$.

Halfspace. Letting $H_\nu^+ = \{x \in \mathbb{R}^n : x \cdot \nu \geq 0\}$ for $\nu \in \mathbb{S}^{n-1}$, if $x \cdot \nu \neq 0$ then

$$\nabla^\alpha \chi_{H_\nu^+}(x) = \frac{2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{1-\alpha}{2}\right)} \frac{1}{|x \cdot \nu|^\alpha} \nu.$$

As a consequence, $\mathcal{F}^\alpha H_\nu^+ = \mathbb{R}^n$ and $\nu_{H_\nu^+}^\alpha \equiv \nu$.

Density estimates

Now we go back to sets with **locally finite distributional fractional perimeter**.

The fractional reduced boundary allows for local upper density estimates.

Theorem (Comi-S, 2019)

If $\chi_E \in BV_{loc}^{\alpha, \infty}(\mathbb{R}^n)$ and $x \in \mathcal{F}^\alpha E$, then there exists $r_x > 0$ such that

$$|D^\alpha \chi_E|(B_r(x)) \leq A_{n,\alpha} r^{n-\alpha} \quad \text{and} \quad |D^\alpha \chi_{E \cap B_r(x)}|(\mathbb{R}^n) \leq B_{n,\alpha} r^{n-\alpha}$$

for all $r \in (0, r_x)$, where $A_{n,\alpha}, B_{n,\alpha} > 0$ are universal constants.

Integration-by-parts on balls (Comi-S, 2019)

If $\chi_E \in BV_{loc}^{\alpha, \infty}(\mathbb{R}^n)$ and $x \in \mathcal{F}^\alpha E$, then for all $\varphi \in \text{Lip}_c(\mathbb{R}^n)$ and a.e. $r > 0$

$$\int_{E \cap B_r(x)} \text{div}^\alpha \varphi \, dy + \int_E \varphi \cdot \nabla^\alpha \chi_{B_r(x)} \, dy + \int_E \text{div}_{\mathbb{N}}^\alpha (\chi_{B_r(x)}, \varphi) \, dy = - \int_{B_r(x)} \varphi \cdot D^\alpha \chi_E.$$

By a standard covering argument, we thus get that

$$|D^\alpha \chi_E| \leq C_{n,\alpha} \mathcal{H}^{n-\alpha} \llcorner \mathcal{F}^\alpha E \quad \text{so that} \quad \dim_{\mathcal{H}}(\mathcal{F}^\alpha E) \geq n - \alpha.$$

Recall that $\dim_{\mathcal{H}}(\mathcal{F}^\alpha E) = n$ in the $W^{\alpha,1}$ regime (in particular, for BV sets)!

Fractional De Giorgi's Blow-up Theorem

The **blow-ups** at $x \in \mathbb{R}^n$ of a set $E \subset \mathbb{R}^n$ are the family

$$\text{Tan}(E, x) = \left\{ \text{limit points of } \left(\frac{E-x}{r} \right)_{r>0} \text{ in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ as } r \rightarrow 0^+ \right\}.$$

We can prove a fractional analogue of De Giorgi's Blow-up Theorem.

Theorem (Comi-S, 2019)

If $\chi_E \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^n)$ and $x \in \mathcal{F}^\alpha E$, then:

- existence: $\text{Tan}(E, x) \neq \emptyset$;
- rigidity: $F \in \text{Tan}(E, x) \implies \nu_F^\alpha(y) = \nu_E^\alpha(x)$ for $|D^\alpha \chi_F|$ -a.e. $y \in \mathcal{F}^\alpha F$.

Open Problem: How to characterize blow-ups?

Bad News Theorem (Comi-S, 2019-22)

The **coarea formula** and the **local chain rule** do **NOT** hold!

$$|D^\alpha f| \neq \int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}| dt \quad \text{and} \quad |D^\alpha \Phi(f)| \neq |\Phi'| |D^\alpha f|$$

Further properties of blow-ups

Let $\chi_E \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^n)$ and $x \in \mathcal{F}^\alpha E$ and assume that $\nu_E^\alpha(x) = \vec{e}_n$.

Theorem (Comi-S, 2023)

If $F \in \text{tan}(E, x)$, then $F = \mathbb{R}^{n-1} \times M$ with $M \subset \mathbb{R}$ such that

- $\chi_M \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R})$ with $\partial^\alpha \chi_M \geq 0$;
- $|M|, |M^c| \in \{0, +\infty\}$;
- $|M| = +\infty \implies \text{ess sup } M = +\infty$;
- $M \neq \emptyset, \mathbb{R}$ is such that $P(M) < +\infty \implies M = (m, +\infty)$ for some $m \in \mathbb{R}$.

Consequence: halfspaces and \mathbb{R}^n are the **ONLY** blow-up cones!

Theorem (Comi-S, 2023)

Assume $f \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^n)$.

- $D_i^\alpha f = 0 \iff D_i f = 0$ whatever $i \in \{1, \dots, n\}$
- $D_1 f = 0 \implies \exists g \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^{n-1})$ such that $f((x_1, x')) = g(x')$ a.e. and
$$(D_{\mathbb{R}^n}^\alpha)_i f = \mathcal{L}^1 \otimes (D_{\mathbb{R}^{n-1}}^\alpha)_i g \quad \text{for } i = 2, \dots, n.$$

Non-local boundaries [1/2]

$$\chi_E \in BV_{loc}^{\alpha, \infty}(\mathbb{R}^n) \implies D^\alpha \chi_E = D_{ac}^\alpha \chi_E + D_S^\alpha \chi_E, D_{ac}^\alpha \chi_E \ll \mathcal{L}^n, D_S^\alpha \chi_E \perp \mathcal{L}^n$$

Theorem (Schönberger, 2023; Comi-S, 2023)

Locality of singular part: $D_S^\alpha \chi_{E \cap F} = \chi_{F^1} D_S^\alpha \chi_E$ whenever $P_\alpha(F) < +\infty$

Notation: $F^t = \left\{ x \in \mathbb{R}^n : \exists \lim_{r \rightarrow 0^+} \frac{|F \cap B_r(x)|}{|B_r(x)|} = t \right\}$ whenever $t \in [0, 1]$.

Theorem (Comi-S, 2023)

$D_S^\alpha \chi_E(F^1) = 0$ whenever $\chi_F \in W^{\alpha, 1}(\mathbb{R}^n)$ with either $|E \cap F| = 0$ or $|E^c \cap F| = 0$

Notation: $\partial^- E = \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < |B_r(x)| \text{ for all } r > 0\}$.

Corollary: $\text{supp}(|D_S^\alpha \chi_E|) \subset \partial^- E$ and so $|D_S^\alpha \chi_E| \leq C_{n, \alpha} \mathcal{H}^{n-\alpha} \llcorner (\mathcal{F}^\alpha E \cap \partial^- E)$

Analogies with classical results [De Giorgi's Theory & Lombardini, 2019]

- $\chi_E \in BV_{loc}(\mathbb{R}^n) \implies \text{supp}(|D\chi_E|) = \partial^- E$
- $\chi_E \in W_{loc}^{\alpha, 1}(\mathbb{R}^n) \implies \partial^- E = \{x \in \mathbb{R}^n : [\chi_E]_{W^{\alpha, 1}(B_r(x))} > 0 \text{ for all } r > 0\}$

Caution: there exists $E \subset \mathbb{R}^n$ such that $P(E) < +\infty$ and $\mathcal{L}^n(\partial^- E) = +\infty!$

Non-local boundaries [2/2]

Exercise (measure-theoretic interior and exterior)

If $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is measurable, then

$$\text{Tan}(E, x) = \{\mathbb{R}^n\} \iff x \in E^1 \quad \text{and} \quad \text{Tan}(E, x) = \{\emptyset\} \iff x \in E^0$$

Notation: $\partial^* E = \mathbb{R}^n \setminus (E^0 \cup E^1)$ is the **measure-theoretic boundary**.

Definition (Effective fractional reduced boundary)

We define $\mathcal{F}_e^\alpha E = \mathcal{F}^\alpha E \cap \partial^* E$ whenever $\chi_E \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^n)$.

Theorem (Comi-S, 2023)

- $\chi_E \in BV_{\text{loc}}^\alpha(\mathbb{R}^n)$, $x \in \mathcal{F}_e^\alpha E$, $\text{Tan}(E, x) = H_\nu^+$ for $\nu \in \mathbb{S}^{n-1} \implies \nu_E^\alpha(x) = \nu$
- $\chi_E \in W_{\text{loc}}^{\alpha, 1}(\mathbb{R}^n) \implies \mathcal{H}^{n-\alpha}(\mathcal{F}_e^\alpha E) = 0$
- $\chi_E \in BV_{\text{loc}}(\mathbb{R}^n) \implies \mathcal{F}E \subset \mathcal{F}_e^\alpha E$, $\mathcal{H}^{n-1}(\mathcal{F}_e^\alpha E \setminus \mathcal{F}E) = 0$, $\nu_E^\alpha = \nu_E$ on $\mathcal{F}E$

Asymptotics

Now important: $\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \sim \frac{1-\alpha}{|B_1|}$ as $\alpha \rightarrow 1^-$.

Asymptotics for $\alpha \rightarrow 1^-$ (Comi-S, 2019) \rightarrow Bourgain-Brezis-Mironescu

- $f \in W^{1,p}(\mathbb{R}^n) \implies \nabla^\alpha f \rightarrow \nabla f$ in L^p whenever $p \in [1, +\infty)$.
- $f \in BV(\mathbb{R}^n) \implies D^\alpha f \rightarrow Df, |D^\alpha f| \rightarrow |Df|, |D^\alpha f|(\mathbb{R}^n) \rightarrow |Df|(\mathbb{R}^n)$.

Asymptotics for $\alpha \rightarrow 0^+$ (Bruè-Calzi-Comi-S, 2020) \rightarrow Maz'ya-Shaposhnikova

- $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n) \implies \nabla^\alpha f \rightarrow Rf$ in L^p whenever $p \in (1, +\infty)$.
- $f \in H^1(\mathbb{R}^n) \cap \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n) \implies \nabla^\alpha f \rightarrow Rf$ in L^1 and in H^1 .
- $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n) \implies \alpha \int_{\mathbb{R}^n} |\nabla^\alpha f(x)| dx \rightarrow c_n \left| \int_{\mathbb{R}^n} f(x) dx \right|$.

Notation: The **Hardy space** is $H^1 = \{f \in L^1(\mathbb{R}^n) : Rf \in L^1(\mathbb{R}^n)\}$ (with $R = \nabla^0$).

Update: **distributional BBM formula** obtained by [Brezis-Mironescu, April 2023]!

Fractional interpolation inequalities

We prove $\nabla^\alpha \rightarrow R$ strongly as $\alpha \rightarrow 0^+$ via fractional interpolation inequalities.

Theorem (Bruè-Calzi-Comi-S, 2020)

Let $\alpha \in (0, 1]$. There exists $c_{n,\alpha} > 0$ such that

$$|D^\beta f|(\mathbb{R}^n) \leq c_{n,\alpha} \|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} |D^\alpha f|(\mathbb{R}^n)^{\frac{\beta}{\alpha}}$$

for all $\beta \in [0, \alpha]$ and all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$.

Theorem (Bruè-Calzi-Comi-S, 2020)

Let $p \in (1, +\infty)$. There exists $c_{n,p} > 0$ such that

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)} \leq c_{n,p} \|\nabla^\gamma f\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \leq \gamma \leq \beta \leq \alpha \leq 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$. There exists $c_n > 0$ such that

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)} \leq c_n \|\nabla^\gamma f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \leq \gamma \leq \beta \leq \alpha \leq 1$ and all $f \in HS^{\alpha,1}(\mathbb{R}^n)$.

$HS^{\alpha,1}(\mathbb{R}^n) = \{f \in H^1(\mathbb{R}^n) : \nabla^\alpha f \in H^1(\mathbb{R}^n; \mathbb{R}^n)\}$ is the **Hardy-Sobolev space**.

Γ -convergence

Definition (Γ -convergence after De Giorgi)

We say that $F_k: (X, d) \rightarrow \overline{\mathbb{R}}$ Γ -converge to $F: (X, d) \rightarrow \overline{\mathbb{R}}$ if

- $x_k \rightarrow x \implies F(x) \leq \liminf_k F(x_k)$
- $\forall x \in X \exists x_k \rightarrow x$ such that $F(x) \geq \limsup_k F(x_k)$

Γ -convergence as $\alpha \rightarrow 1^-$ (Comi-S, 2022) \rightarrow Ambrosio-De Philippis-Martinazzi

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary or $\Omega = \mathbb{R}^n$.

- $f \in BV(\Omega) \implies \Gamma(L^1) - \lim_{\alpha \rightarrow 1^-} |D^\alpha f|(\Omega) = |Df|(\Omega)$
- $E \subset \mathbb{R}^n$ measurable $\implies \Gamma(L^1_{loc}) - \lim_{\alpha \rightarrow 1^-} |D^\alpha \chi_E|(\Omega) = P(E; \Omega)$

Why is Γ -convergence interesting? Minimizers of F_k converge to minimizers of F !

Possible application: regularity of **distributional** non-local minimal surfaces.

Update: **similar Γ -convergence results** obtained by [Brezis-Mironescu, April 2023]!

Remark: asymptotics and Γ -convergence can be generalized to $\alpha \rightarrow \alpha_0^- \in (0, 1]$.

Open problems and research directions

About sets and perimeter

- ▷ Is there a set with \mathcal{L}^n -negligible fractional reduced boundary?
- ▷ What are 1D profiles of **blow-ups**?
- ▷ Are balls **isoperimetric** sets for the fractional variation?
- ▷ What about **minimal surfaces** for the fractional variation (existence, regularity)?

About functions and variation

- ▷ Do $BV^{\alpha,p}$ functions satisfy some good **local** properties?
 - precise representatives? (SPOILER)
 - absolute continuity properties of the fractional variation? (SPOILER)
 - approximate limits? jumps?
- ▷ Can $BV^{\alpha,p}$ functions be defined **on a general open set** $\Omega \subset \mathbb{R}^n$?

Thank you for your attention!