Non-local BV functions and a denoising model with L^1 fidelity

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Shape Optimization, Geometric Inequalities, and Related Topics

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What are denoising models?

In image processing, denoising preserves the most significant features of an image while removing the background noise.

Original | Noisy image | Denoised image

Source: Wikipedia

Total variation denoising models

Setting: screen $\rightsquigarrow \mathbb{R}^n$, source image (corrupted) $\rightsquigarrow f$, final image (denoised) $\rightsquigarrow u$.

$$\min_{u \in BV(\mathbb{R}^n)} [u]_{BV} + \frac{\Lambda}{p} \int_{\mathbb{R}^n} |u - f|^p \, dx$$

where $p \in [1, \infty)$ and $\Lambda > 0$ is the fidelity.

Applications: gravitational-waves (2018) and black hole in Messier 87 galaxy (2019)

Important models: ROF vs CE

$$\min_{u \in BV(\mathbb{R}^n)} [u]_{BV} + \frac{\Lambda}{p} \int_{\mathbb{R}^n} |u - f|^p \, dx$$

$p = 2 \rightsquigarrow$ Rudin-Osher-Fatemi (ROF) model (1992)

- preserves sharp discontinuities (edges), removes fine scale details
- allows for discontinuities, disfavors large oscillations
- strictly convex, hence uniqueness of minimizer $u = u(f, \Lambda)$
- NOT contrast invariant: u solution for f, cu not solution for cf with $c > 0$

$p = 1 \rightsquigarrow$ Chan and Esedoḡlu (CE) model (2005)

- contrast invariant
- convex but NOT strictly, hence non-uniqueness of minimizers
- depends on the shape of the images
- level-set decoupling via coarea formula

$$[u]_{BV} = \int_{\mathbb{R}} P(\{u > t\}) \, dt$$

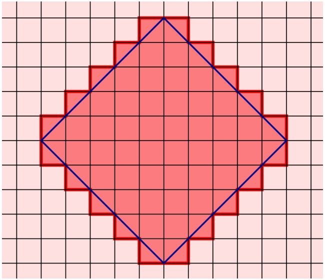
The importance of total variation: local vs non-local

Local BV

- quite efficient in reducing the noise and reconstructing the main features
- scarcely preserves the details and textures of the datum

Non-local BV

- good for digital images/filters
- weights the affinity between different parts/pixels in the image
- considers both geometric parts and textures



Source: Dipierro-Valdinoci (2018)

What is non-local variation?

Keep in mind: non-local = 'distant points count'

Non-local total variation with kernel K

$$[u]_{BV^K} = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)| \, K(x - y) \, dx \, dy$$

where $K \geq 0$ is a kernel

Important examples:

- $K \in L^1(\mathbb{R}^n)$ gives $[u]_{BV^K} \leq \|K\|_{L^1} \|u\|_{L^1}$ [Mazón-Solera-Toledo]
- $K(x) = \frac{1}{|x|^{n+s}}$ gives the Gagliardo-Slobodeckij-Sobolev seminorm for $p = 1$,

$$[u]_{W^{s,p}} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p} \quad \text{for } p \in [1, \infty)$$

[Bessas], [Bessas-S.], [Novaga-Onoue]

Others: [Buades-Coll-Morel], [Kindermann-Osher-Jones], [Gilboa-Osher], [Antil-Dı́ıaz-Jing-Schikorra] using [Comi-S.] and more...

Plan

STEP 0. We choose a kernel $K \geq 0$ and define the (non-local total) K-variation

$$[u]_{BV^K} = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)| \, K(x-y) \, dx \, dy.$$

STEP 1. We study the fundamental properties of the space

$$BV^K(\mathbb{R}^n) = \left\{ u \in L^1(\mathbb{R}^n) : [u]_{BV^K} < \infty \right\}.$$

STEP 2. We use the theory of BV^K functions to study the L^1-denoising model

$$\min_{u \in BV^K(\mathbb{R}^n)} [u]_{BV^K} + \Lambda \int_{\mathbb{R}^n} |u - f| \, dx$$

STEP 3. We study the associated non-local Cheeger problem.

The space of functions with finite K-variation

Let $K \geq 0$ be a kernel on $\mathbb{R}^n$. We focus on non-integrable kernels $K \notin L^1(\mathbb{R}^n)$ only.

The non-local K-variation of $u \in L^1_{loc}(\mathbb{R}^n)$ is

$$[u]_K = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)| \, K(x-y) \, dx \, dy,$$

so $BV^K(\mathbb{R}^n) = \left\{ u \in L^1(\mathbb{R}^n) : [u]_K < \infty \right\}$. The K-perimeter is $P_K(E) = [\chi_E]_K$.

Basic properties

- isometries: $[\,\cdot\,]_K$ is translation invariant, homogeneous and $[c]_K = 0$
- min-max: $[u \wedge v]_K + [u \vee v]_K \leq [u]_K + [v]_K$
- Fatou: $u_k \to u$ in $L^1_{loc}(\mathbb{R}^n) \implies [u]_K \leq \liminf_k [u_k]_K$
- coarea formula: $[u]_K = \displaystyle\int_{\mathbb{R}} P_K(\{u > t\}) \, dt$
- $BV \subset BV^K$: $[u]_K \leq \max\left\{ \|u\|_{L^1}, \tfrac{1}{2}[u]_{BV} \right\} \displaystyle\int_{\mathbb{R}^n} (1 \wedge |x|) \, K(x) \, dx$

Sequential compactness in $W^{K,p}$

To prove existence of minimizers $u = u(f, \Lambda)$, we need compactness in BV^K.

We work in the more general space $W^{K,p} \subset L^p$ with $p \in [1, +\infty)$ and

$$[u]_{K,p} = \left(\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^p \, K(x - y) \, dx \, dy \right)^{1/p}$$

For $p = 1$ we recover $[u]_{K,1} = [u]_{BV^K}$ and $W^{K,1} = BV^K$.

Sequential compactness [Bessas-S.], [Foghem Gounoue in Ph.D. thesis]

$$K \notin L^1(\mathbb{R}^n), \quad K \in L^1(\mathbb{R}^n \setminus B_r) \text{ for all } r > 0$$

$$\Downarrow$$

$$(u_h)_h \subset W^{K,p} \text{ bounded} \implies \exists \text{ subsequence } (u_{h_j})_j \; L^p_{loc}\text{-converging to } u \in W^{K,p}$$

Idea of proof: $T_\eta(u) = u * \eta$ is $L^p \to L^p$ locally compact for $\eta \in L^1$ and

$$\|u - T_{\eta_\delta}(u)\|_{L^p} \lesssim \|K_\delta\|^{-1/p} [u]_{K,p}$$

for $\eta_\delta = K_\delta / \|K_\delta\|_{L^1}$ and $K_\delta = K \, \mathbf{1}_{\mathbb{R}^n \setminus B_\delta}$. Note that $\|K_\delta\|_{L^1} \to \infty$ as $\delta \to 0^+$. □

Isoperimetric inequality

For $v > 0$, we let $B^v = B_{r_v}$ with $r_v = (v/|B_1|)^{1/n}$, so that $|B^v| = v$.

Isoperimetric inequality [Bessas-S.], [Cesaroni-Novaga], [De Luca-Novaga-Ponsiglione]

$$K \text{ radially symmetric decreasing } \implies P_K(E) \geq P_K(B^{|E|}), \text{ with } |B^{|E|}| = |E|$$

$$\text{equality } \iff E \text{ is a ball, if } K \text{ radial}^+ \text{ in a ngbh of the origin}$$

Idea of proof: Apply Riesz rearrangement inequality to

$$P_K(E) = \int_0^{\|K\|_{L^\infty}} |E|\,|\{K > t\}| - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_E(x)\,\mathbf{1}_E(y)\,\mathbf{1}_{\{K>t\}}(x-y)\,dx\,dy\,dt$$

noticing that $\{K > t\} = B_{R(t)}$ is a ball for some $R(t) \in [0, \infty]$. □

Open problem: find isoperimteric sets for $K \notin L^1$ NOT radially symmetric!

Corollary [Bessas-S.]

$$K \text{ radially symmetric decreasing } \implies [u]_K \geq [u^\star]_K$$

$$\text{equality } \iff u \geq 0,\ \{u > t\} \text{ is a ball, if } K \text{ radial}^+ \text{ in a ngbh of the origin}$$

where $u^\star$ is the symmetric decreasing rearrangement of u (apply coarea formula).

Monotonicity formula

Assume K is q-decreasing: $|x| \leq |y| \implies K(x)|x|^q \geq K(y)|y|^q$ for $q \geq 0$.

Fun fact: q-decreasing for $q \geq n+1 \implies BV^K$ functions are constant! [Brezis]

Monotonicity [Bessas-S.]: $0 < r \leq R < +\infty \implies \dfrac{P_K(rE)}{|rE|^{2-\frac{q}{n}}} \geq \dfrac{P_K(RE)}{|RE|^{2-\frac{q}{n}}}$

Idea of proof: Observe that (for simplicity, K is symmetric)

$$P_K(\lambda E) = \int_{\lambda E}\int_{(\lambda E)^c} K(x-y)\,dx\,dy = \lambda^{2n}\int_E\int_{E^c} K(\lambda(x-y))\,dx\,dy$$

for $\lambda > 0$ by changing variables, then apply q-decreasing assumption. □

Isoperimetric inequality for small volumes [Bessas-S.]

K radial and $q < n+1$: $\quad |E| \leq |B| \implies \dfrac{P_K(E)}{|E|^{2-\frac{q}{n}}} \geq \dfrac{P_K(B)}{|B|^{2-\frac{q}{n}}}$

Gagliardo-Nirenberg-Sobolev for finite support [Bessas-S.]

$u \in BV^K$ with $|\text{supp}(u)| < \infty \implies \|u\|_{L^{\frac{n}{2n-q},1}} \leq C^{iso}_{n,q,|\text{supp}(u)|}[u]_K$.

Intersection with convex sets

Assume K is radial, 1-decreasing and $\int_{\mathbb{R}^n} (1 \wedge |x|)\, K(x)\, dx < \infty$.

Intersection with convex sets [Bessas-S.]

$$|E| < \infty \implies P_K(E \cap C) \leq P_K(E) \text{ for all } C \subset \mathbb{R}^n \text{ convex}$$

Idea of proof [Figalli-Fusco-Maggi-Millot-Morini]: After reducing to $C = H$ half-space and E bounded, for $x_0 \in \partial H$ and $B_R(x_0) \supset E$ one can estimate

$$P_K(E) - P_K(E \cap H) \geq P_K(F; B_R(x_0)) - P_K(H; B_R(x_0))$$

where $F = E \cup H$ and

$$P_K(F; A) = \left(\int_{E \cap A} \int_{E^c \cap A} + \int_{E \cap A} \int_{E^c \cap A^c} + \int_{E \cap A^c} \int_{E^c \cap A} \right) K(x - y)\, dx\, dy$$

is the K-perimeter of F relative to A. The conclusion follows from

Local minimality of half-spaces [Pagliari], [Cabré]

$$H \text{ is a half-space, } 0 \in \partial H \implies P_K(H; B_R) \leq P_K(E; B_R) \text{ if } E \setminus B_R = H \setminus B_R$$

K-Archimedes: $A \subset B$ with A convex and $|B| < +\infty \implies P_K(A) \leq P_K(B)$

Functional K-variation denoising problem with L^1 fidelity

<u>Data</u>: $\mathbb{R}^n$ screen, $f \in L^1_{\text{loc}}$ corrupted image and $\Lambda > 0$ fidelity.

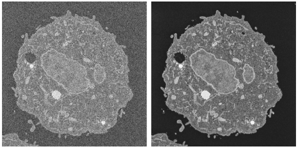
We study the functional K-variation L^1 denoising problem

$$\text{(FP)} \qquad \min_{u \in L^1_{\text{loc}}(\mathbb{R}^n)} [u]_{BV^K} + \Lambda \int_{\mathbb{R}^n} |u - f| \, d\nu$$

where $\nu \in \mathcal{W}(\mathbb{R}^n) = \{\nu = w\mathscr{L}^n : w \in L^\infty, \inf_{\mathbb{R}^n} w > 0\}$ an L^∞-weight measure.

Why L^∞-weight measures? $\rightsquigarrow$ deep learning

- do not alter the L^1 nature of the approximation term
- more flexibility, adding a degree of freedom in the fidelity
- $\Lambda > 0$ keeps its role of global Lagrangian multiplier
- ν secondary local fidelity parameter (emphasis on specific regions only)



Source: Sun-Parwani

Existence and basic properties for (FP)

Call FSol(f, Λ, ν) the set of solutions of the functional problem

$$\text{(FP)} \qquad \min_{u \in L^1_{\text{loc}}(\mathbb{R}^n)} [u]_{BV^K} + \Lambda \int_{\mathbb{R}^n} |u - f| \, d\nu$$

Existence for (FP) [Bessas-S.]

$K \notin L^1(\mathbb{R}^n)$, $K \in L^1(\mathbb{R}^n \setminus B_r)$ for all $r > 0 \implies$ FSol$(f, \Lambda, \nu) \neq \emptyset$ for $f \in L^1(\mathbb{R}^n)$

Idea of proof: Use lsc of energy and compactness in BV^K. □

Basic properties of F-solutions

- FSol$(f, \Lambda, \nu) \subset L^1_{\text{loc}}$ is convex and closed
- $u_j \in$ FSol(f_j, Λ, ν), $f_j \to f$ in L^1, $u_j \to u$ in $L^1_{\text{loc}} \implies u \in$ FSol(f, Λ, ν)
- FSol$(f + c, \Lambda, \nu) =$ FSol$(f, \Lambda, \nu) + c$
- FSol$(cf, \Lambda, \nu) = c\,$FSol(f, Λ, ν)
- $u \in$ FSol$(f, \Lambda, \nu) \implies u^+ \in$ FSol(f^+, Λ, ν), $u^- \in$ FSol(f^-, Λ, ν)
- $u \in$ FSol$(f, \Lambda, \nu) \implies u \wedge c \in$ FSol$(f \wedge c, \Lambda, \nu)$, $u \vee c \in$ FSol$(f \vee c, \Lambda, \nu)$

Geometric K-variation denoising problem with L^1 fidelity

We also study the geometric K-variation L^1 denoising problem ($f = \chi_E$, $u = \chi_U$)

$$(\mathrm{GP}) \qquad \min_{U \subset \mathbb{R}^n} P_K(U) + \Lambda \nu(U \triangle E)$$

and we let $\mathrm{GSol}(E, \Lambda, \nu)$ be set of solutions to the geometric problem.

Basic properties of G-solutions

- $U \in \mathrm{GSol}(E, \Lambda, \nu) \implies U + x \in \mathrm{GSol}(E + x, \Lambda, \nu_x)$, $\nu_x(A) = \nu(A - x)$
- $U_j \in \mathrm{GSol}(E_j, \Lambda, \nu)$, $E_j \to E$ in L^1, $U_j \to U$ in $L^1_{loc} \implies U \in \mathrm{GSol}(E, \Lambda, \nu)$
- $U \in \mathrm{GSol}(E, \Lambda, \nu) \implies U^c \in \mathrm{GSol}(E^c, \Lambda, \nu)$
- $\mathrm{GSol}(E, \Lambda, \nu)$ closed under finite intersection and finite union
- $\mathrm{GSol}(E, \Lambda, \nu)$ closed under count. decr. intersection and count. incr. union

Relation between F-solutions and G-solutions

- $u \in \mathrm{FSol}(f, \Lambda, \nu) \implies \{u > t\} \in \mathrm{GSol}(\{f > t\}, \Lambda, \nu)$ for all $t \in \mathbb{R} \setminus \{0\}$
- $\{u > t\} \in \mathrm{GSol}(\{f > t\}, \Lambda, \nu)$ for a.e. $t \in \mathbb{R} \implies u \in \mathrm{FSol}(f, \Lambda, \nu)$

Moreover, if $|E| < \infty$, then:

- $U \in \mathrm{GSol}(E, \Lambda, \nu) \implies \chi_U \in \mathrm{FSol}(\chi_E, \Lambda, \nu)$
- $u \in \mathrm{FSol}(\chi_E, \Lambda, \nu) \implies 0 \le u \le 1$ a.e., $\{u > t\} \in \mathrm{GSol}(E, \Lambda, \nu)$ for $t \in (0, 1)$

Existence for (GP) and basic properties

Existence for (GP) [Bessas-S.]

$K \notin L^1(\mathbb{R}^n)$, $K \in L^1(\mathbb{R}^n \setminus B_r)$ for all $r > 0 \implies \mathrm{GSol}(E, \Lambda, \nu) \neq \emptyset$ for $|E| < \infty$

Bounded geometric datum [Bessas-S.]

Assume K radial, 1-decreasing and $\int_{\mathbb{R}^n} (1 \wedge |x|)\, K(x)\, dx < \infty$.

(1) $E \subset B_R \implies U \subset B_R$ for all $U \in \mathrm{GSol}(E, \Lambda, \nu)$

Moreover, if also $K \notin L^1$, then

(2) E bounded convex $\implies \mathrm{FSol}(\chi_E, \Lambda, \nu) = \{\chi_{U_\Lambda}\}$ for a.e. $\Lambda > 0$ with $U_\Lambda \subset E$

Idea of proof:

(1) $\nu((U \cap B_R) \cap E) \leq \nu(U \cap E)$ and $P_K(U \cap B_R) \leq P_K(U)$, since B_R convex.

(2) Consider monotone maps $\Lambda \mapsto \inf / \sup\{\|u - \chi_E\|_{L^1(\nu)} : u \in \mathrm{FSol}(\chi_E, \Lambda, \nu)\}$.

Prove that $\mathrm{FSol}(\chi_E, \Lambda, \nu) = \{u_\Lambda\}$ for $\Lambda > 0$ outside countable jump set.

Observe that $u = \chi_U$ for some $U \subset E$ by basic properties.

Since $\mathrm{FSol}(\chi_E, \Lambda, \nu)$ is convex, U is unique. □

Maximal and minimal solutions of (GP)

Existence of max and min solutions of (GP) [Bessas-G.]

Assume $K \notin L^1(\mathbb{R}^n)$, $K \in L^1(\mathbb{R}^n \setminus B_r)$ for all $r > 0$. If $|E| < \infty$, then (GP) admits a minimal and a maximal solution $E^-, E^+ \in \mathrm{GSol}(E, \Lambda, \nu)$ w.r.t. inclusion.

Properties: $E^- \subset E^+$, $(E^c)^- = (E^+)^c$, $(E^c)^+ = (E^-)^c$, $\nu(E^-) \leq \nu(E^+) \leq 2\nu(E)$.

Idea of proof: To construct E^-, choose a minimizing sequence for

$$\inf\{\nu(U) : U \in \mathrm{GSol}(E, \Lambda, \nu)\} \in [0, 2\nu(E)]$$

and then use closure w.r.t. finite and countable decreasing intersection.
Argue analogously for constructing E^+. □

Comparison principle for (GP) [Bessas-G.]

Assume $K \notin L^1(\mathbb{R}^n)$, $K \in L^1(\mathbb{R}^n \setminus B_r)$ for all $r > 0$, K symmetric and $K > 0$.
If $P_K(E_i) < \infty$ and $\min\{|E_i|, |E_i^c|\} < \infty$ for $i = 1, 2$, then

$$E_2 \subset E_1 \implies (E_2)^- \subset (E_1)^- \text{ and } (E_2)^+ \subset (E_1)^+$$

Proof is tricky! One compares $U_1 \in \mathrm{GSol}(E_1, \Lambda, \nu)$ with $U_2 \in \mathrm{GSol}(E_2, \Lambda, \nu)$.

Remark $K > 0$ can be weaken to get a comparison principle at small scales.

High fidelity

When fidelity $\Lambda > 0$ is high, the solution $u = u(f, \Lambda)$ is very close to the datum f.

Assume K radial$^+$, $\int_{\mathbb{R}^n} (1 \wedge |x|)\, K(x)\, dx < \infty$, $K \notin L^1$, K 1-decreasing and $K > 0$.

High fidelity for $C^{1,1}$ regular sets [Bessas-S.]

Let E be $C^{1,1}$ regular open set with $\min\{|E|, |E^c|\} < \infty$. There is $\bar{\Lambda} > 0$ such that

$$\mathrm{GSol}(E, \Lambda, \nu) = \{E\} \quad \text{and} \quad \mathrm{GSol}(E^c, \Lambda, \nu) = \{E^c\} \quad \text{for all } \Lambda > \bar{\Lambda}.$$

Idea of proof: By $C^{1,1}$ regularity and comparison, we reduce to $E = B_R(x)$ a ball.

In this case, $\mathrm{GSol}(B_R(x), \Lambda, \nu) = \{B_r(x)\}$ by isoperimetric inequality for $0 \le r \le R$.

To prove $r = R$, one exploits the monotonicity of P_K (K is 1-decreasing). □

Arguing via level sets, one can extend the previous result to functions.

High fidelity for uniformly $C^{1,1}$ regular functions [Bessas-S.]

Let $f \in L^1$ have uniformly $C^{1,1}$ regular superlevel sets. There is $\bar{\Lambda} > 0$ such that

$$\mathrm{FSol}(f, \Lambda, \nu) = \{f\} \quad \text{for all } \Lambda > \bar{\Lambda}.$$

uniformly $C^{1,1}$ regular superlevels = inner/outer radius of $\{f > t\}$ uniform in $t \in \mathbb{R}$

Low fidelity

When fidelity $\Lambda > 0$ is low, the solution $u = u(f, \Lambda)$ is very close to black screen.

Assume K radial, $\int_{\mathbb{R}^n} (1 \wedge |x|)\, K(x)\, dx < \infty$, $K \notin L^1$, K 1-decreasing and

$$K \text{ } D\text{-doubling: } \exists\, C > 0 \text{ s.t. } |y| = 2|x|,\ |x| \le 2D \implies K(x) \le C\, K(y)$$

Low fidelity [Bessas-S.]

For $R < D/4$ there is $\bar{\Lambda} > 0$ such that

$$f \in L^1 \text{ with } \operatorname{supp}(f) \subset B_R \implies \mathrm{FSol}(f, \Lambda, \nu) = \{0\} \text{ for all } \Lambda < \bar{\Lambda}.$$

Idea of proof: First reduce to $f \ge 0$ and so $u \ge 0$. By minimality

$$[u]_K + \Lambda \|u - f\|_{L^1(B_R, \nu)} \le \Lambda \|f\|_{L^1(B_R, \nu)}.$$

The trick is to estimate $[u]_K \gtrsim_h \|u(\cdot + h) - u\|_{L^1} = 2\|u\|_{L^1} \gtrsim_\nu \|u\|_{L^1(B_R, \nu)}$ for $2R \le |h| \le \frac{D}{2}$. The first inequality follows from an L^1-estimate on translation of BV^K functions which, in turn, is a consequence of a pointwise Lusin-type estimate

$$|u(x) - u(y)| \le \omega_{K,D}(|x - y|)\, \big(\mathbf{D}_K u(x) + \mathbf{D}_K u(y)\big),$$

$$\mathbf{D}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^n} |u(x) - u(z)|\, K(x - z)\, dz \quad \text{and} \quad \omega_{K,D} \text{ modulus of continuity}$$

The non-local Cheeger problem

Total variation denoising models can be naturally connected with Cheeger problem.

The Cheeger problem for the K-variation in an admissible $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$ is

$$h_{K,\nu}(\Omega) = \inf\left\{ \frac{P_K(E)}{\nu(E)} : E \subset \Omega,\ |E| \in (0,\infty) \right\} \in [0,\infty)$$

We call $h_{K,\nu}(\Omega)$ the Cheeger constant of Ω and any minimizer a Cheeger set of Ω.

Existence of Cheeger sets and basic properties [Bessas-S.]

Let K radial, $K \in L^1(\mathbb{R}^n \setminus B_r)\ \forall r > 0$, $K \notin L^1$ and q-decreasing with $q < n+1$. Cheeger sets E exist (hence $h_{K,\nu}(\Omega) > 0$) with

$$|E|^{\frac{q}{n}-1} \geq C^{iso}_{|\Omega|,n,q,\nu}\, h_{K,\nu}(\Omega).$$

Moreover, $\partial E \cap \partial\Omega \neq \emptyset$ for $\nu = \mathscr{L}^n$, Ω open and K n-decreasing^{+}.

Idea of proof: exploit compactness in BV^K, isoperimetric ineq. and monotonicity. □

Further properties for $\nu = \mathscr{L}^n$ [Bessas-S.]

- calibrability: balls are self-Cheeger sets
- K-Faber-Krahn inequality: $h_K(\Omega) \geq h_K(B^{|\Omega|})$ where $|B^{|\Omega|}| = |\Omega|$

Relation between (GP) and Cheeger problem

Assume K radial, $\int_{\mathbb{R}^n} (1 \wedge |x|)\, K(x)\, dx < \infty$, $K \notin L^1(\mathbb{R}^n)$, K q-decreasing with $q \in [1, n+1)$ and D-doubling with $D = \infty$.

Relation between (GP) and Cheeger problem [Bessas-S.]

Let E be a bounded convex set with non-empty interior.

(1) $h_{K,\nu}(E) = \sup\{\Lambda > 0 : \emptyset \in \mathrm{GSol}(E, \Lambda, \nu)\} \in (0, \infty)$.

(2) $\Lambda < h_{K,\nu}(E) \implies \mathrm{GSol}(E, \Lambda, \nu) = \{\emptyset\}$.

(3) $\Lambda = h_{K,\nu}(E) \implies \mathrm{GSol}(E, \Lambda, \nu) = \mathcal{C}_{K,\nu}(E) \cup \{\emptyset\}$ and so

$$\mathrm{FSol}(\chi_E, h_{K,\nu}(E), \nu) = \left\{u \in BV^K(\mathbb{R}^n; [0,1]) : \{u > t\} \in \mathcal{C}_{K,\nu}(E) \cup \{\emptyset\}\right\}$$

(4) $\Lambda > h_{K,\nu}(E)$ and E is calibrable $\implies \mathrm{GSol}(E, \Lambda, \nu) = \{E\}$.

For $\nu = \mathscr{L}^n$ and $E =$ ball B, such result can be improved as

$$\mathrm{GSol}(B, \Lambda, \mathscr{L}^n) = \begin{cases} \{\emptyset\} & \text{for } \Lambda < \Lambda_0 \\ \{\emptyset, B\} & \text{for } \Lambda = \Lambda_0 \\ \{B\} & \text{for } \Lambda > \Lambda_0 \end{cases} \qquad \text{where } \Lambda_0 = \frac{P_K(B)}{|B|}$$

THANK YOU FOR YOUR ATTENTION!

Slides available via giorgio.stefani.math@gmail.com or giorgiostefani.weebly.com.

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