Bakry-Émery curvature condition and entropic inequalities on metric-measure groups

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Bakry-Émery curvature-dimension condition

Let $(\mathbb{M}, \mathsf{g})$ be a smooth Riemannian manifold with Laplace-Beltrami operator Δ.

The heat flow $f_t = \mathsf{P}_t f$ starting from a datum f is associated to $\partial_t - \Delta$.

Define $\varphi(s) = \mathsf{P}_s \Gamma(\mathsf{P}_{t-s} f)$ for $s \in [0, t]$ and $t > 0$. One can check that

$$\boxed{\varphi'(s) = 2\mathsf{P}_s \Gamma_2(\mathsf{P}_{t-s} f)}$$

where $\Gamma(f, g) = \langle \nabla f, \nabla g \rangle_{\mathsf{g}}$ and $\Gamma_2(f, g) = \frac{1}{2}\big(\Delta\Gamma(f, g) - \Gamma(\Delta f, g) - \Gamma(f, \Delta g)\big)$.

Note that $\boxed{\Gamma_2(f) = ||\mathrm{Hess} f||_2^2 + \mathrm{Ric}(\nabla f, \nabla f),}$ where $\mathrm{Ric}(\cdot, \cdot)$ is the Ricci tensor.

If $\mathrm{Ric}(v, v) \geq K |v|_{\mathsf{g}}^2$ for some $K \in \mathbb{R}$, then $\varphi'(s) \geq 2K\varphi(s)$ and thus

$$\boxed{\mathrm{Ric} \geq K \implies \Gamma(\mathsf{P}_t f) \leq e^{-2Kt} \mathsf{P}_t \Gamma(f)}$$

the Bakry-Émery-Ledoux pointwise gradient estimate for the heat flow. To consider $N = \dim \mathbb{M}$ observe that $\boxed{||\mathrm{Hess} f||_2^2 \geq \frac{1}{N} (\Delta f)^2}$ [Wang, 2011]. Surprisingly, we have an equivalence:

$$\boxed{\mathsf{CD}(K, N) : \mathrm{Ric} \geq K,\ \dim \mathbb{M} \leq N \iff \Gamma_2(f) \geq \frac{1}{N} (\Delta f)^2 + K\,\Gamma(f),}$$

the Bakry-Émery curvature-dimension inequality (we will consider $N = \infty$ only).

The Wasserstein space

We see $(\mathbb{M}, \mathsf{g})$ as a metric space (X, d) with $X = \mathbb{M}$ and $\mathsf{d} = \mathsf{d}_{\mathsf{g}}$.

Theorem (von Renesse - Sturm, 2005)

$$\mathrm{Ric} \geq K \iff W_2(\mathsf{P}_t\mu, \mathsf{P}_t\nu) \leq e^{-Kt} W_2(\mu, \nu) \text{ for all } \mu, \nu \in \mathscr{P}_2(\mathbb{M})$$

Here $(\mathscr{P}_2(X), W_2)$ is the Wasserstein metric space, where

$$\mathscr{P}_2(X) = \left\{ \mu \in \mathscr{P}(X) : \int_X \mathsf{d}(x, x_0)^2 \, \mathrm{d}\mu(x) < +\infty, \; x_0 \in X \right\}$$

and

$$W_2^2(\mu, \nu) = \inf \left\{ \int_{X \times X} \mathsf{d}^2(x, y) \, \mathrm{d}\pi : \pi(x, y) \in \mathrm{Plan}(\mu, \nu) \right\},$$

with

$$\mathrm{Plan}(\mu, \nu) = \{ \pi \in \mathscr{P}(X \times X) : (p_1)_\# \pi = \mu, \; (p_2)_\# \pi = \nu \}.$$

Important fact:

$$(X, \mathsf{d}) \text{ Polish (geodesic)} \implies (\mathscr{P}_2(X), W_2) \text{ Polish (geodesic).}$$

The Boltzmann entropy

As before, we see $(X, \mathsf{d}, \mathfrak{m}) = (\mathbb{M}, \mathrm{g}, \mathrm{Vol}_{\mathrm{g}})$ as a metric-measure space.

Theorem (von Renesse - Sturm, 2005)

$$\mathrm{Ric} \geq K \iff \mathrm{Ent}_{\mathfrak{m}}(\mu_s) \leq (1-s)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2} s(1-s)\, W_2^2(\mu_0, \mu_1)$$

where $s \mapsto \mu_s$ is any (1-speed) W_2-geodesic joining $\mu_0, \mu_1 \in \mathrm{Dom}(\mathrm{Ent}_{\mathfrak{m}})$.

Here $\mathrm{Ent}_{\mathfrak{m}} : \mathscr{P}_2(X) \to (-\infty, +\infty]$ is the (Boltzmann) entropy

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) = \int_X \varrho \log \varrho \, \mathrm{d}\mathfrak{m}$$

for $\mu = \varrho\mathfrak{m} \in \mathscr{P}_2(X)$, with $\mathrm{Ent}_{\mathfrak{m}}(\mu) = +\infty$ if $\mu \not\ll \mathfrak{m}$.

NOTE: we want $\mathrm{Ent}(\mu) > -\infty$ for all $\mu \in \mathscr{P}_2(X)$, but this is OK whenever

$$\exists x_0 \in X \quad \exists A, B > 0 \quad : \quad \mathfrak{m}(B_r(x_0)) \leq A \exp\left(B\, r^2\right) \qquad \text{(exp.ball)}$$

Bishop volume comparison: $(X, \mathsf{d}, \mathfrak{m}) = (\mathbb{M}, \mathrm{g}, \mathrm{Vol}_{\mathrm{g}})$ with $\mathrm{Ric} \geq K \implies$ (exp.ball).

On a smooth Riemannian manifold $(\mathbb{M}, \mathrm{g})$ we know that

(1) $\mathrm{Ric} \geq K$

(2) $\Gamma(\mathsf{P}_t) \leq e^{-2Kt}\,\mathsf{P}_t\Gamma$

(3) $W_2(\mathsf{P}_t, \mathsf{P}_t) \leq e^{-Kt}\,W_2$

(4) $\mathsf{Ent}_{\mathfrak{m}}$ is W_2-geodesic K-convex

are equivalent, but (4) only need d and $\mathfrak{m}$, not the smoothness of $(\mathbb{M}, \mathrm{g})$, hence making sense in metric-measure spaces.

Lott - Villani, Sturm

Definition: $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{CD}(K,\infty)$ space if $\mathsf{Ent}_{\mathfrak{m}}$ is W_2-geodesic K-convex

Natural questions:

What about (2) and (3)?

Can the heat flow be defined in a metric-measure space?

The Cheeger energy

In a metric-measure space $(X, \mathsf{d}, \mathfrak{m})$, the Cheeger energy is

$$\mathsf{Ch}(f) = \inf\left\{ \liminf_n \frac{1}{2}\int_X |\mathsf{D} f_n|^2 \,\mathsf{d}\mathfrak{m} : f_n \to f \text{ in } \mathsf{L}^2(X,\mathfrak{m}),\ f_n \in \mathsf{Lip}(X) \right\}$$

where $|\mathsf{D}f|(x) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{\mathsf{d}(x,y)}$ stands for the slope of $f\colon X \to \mathbb{R}$.

Ch is convex, l.s.c. and its domain $\mathsf{W}^{1,2}(X,\mathsf{d},\mathfrak{m})$ is dense in $L^2(X,\mathfrak{m})$

We can define the heat flow as the (Hilbertian) gradient flow of Ch in $L^2(X,\mathfrak{m})$:

$$\mathsf{P}_t f \underset{t\to 0^+}{\longrightarrow} f \text{ in } \mathsf{L}^2(X,\mathfrak{m}) \quad \text{and} \quad \frac{\mathsf{d}}{\mathsf{d}t}\mathsf{P}_t f \in -\partial^-\mathsf{Ch}(\mathsf{P}_t f) \text{ for a.e. } t > 0.$$

The Laplacian $-\Delta_{\mathsf{d},\mathfrak{m}} f \in \partial^- \mathsf{Ch}(f)$ is the element of minimal $\mathsf{L}^2(X,\mathfrak{m})$-norm.

CAUTION: $\mathsf{W}^{1,2}(X,\mathsf{d},\mathfrak{m})$ with $\|\cdot\|_{\mathsf{W}^{1,2}} = \sqrt{\|\cdot\|^2_{\mathsf{L}^2} + \mathsf{Ch}(\cdot)}$ may be NOT Hilbert!

Example: consider $(\mathbb{R}^n, \|\cdot\|_p, \mathscr{L}^n)$ for $p \neq 2$.

The weak gradient

Any $f \in \mathsf{W}^{1,2}(X, \mathsf{d}, \mathfrak{m})$ has a (unique) weak gradient $|\mathsf{D}f|_w \in \mathsf{L}^2(X, \mathfrak{m})$ such that

$$\mathsf{Ch}(f) = \frac{1}{2} \int_X |\mathsf{D}f|_w^2 \, \mathrm{d}\mathfrak{m}.$$

The weak gradient $|\mathsf{D}f|_w$ behaves like the 'modulus of the gradient' and one can develop Calculus rules in a non-smooth setting.

We say that Ch is quadratic if it satisfies the parallelogram law

$$\mathsf{Ch}(f+g) + \mathsf{Ch}(f-g) = 2\mathsf{Ch}(f) + 2\mathsf{Ch}(g).$$

We assume that Ch is quadratic, so that

$$\mathsf{W}^{1,2}(X, \mathsf{d}, \mathfrak{m}) \text{ is Hilbert}, \quad \mathsf{P}_t \text{ is linear}, \quad \Gamma(f) = |\mathsf{D}f|_w^2 \text{ is quadratic}.$$

By polarization, we can define

$$\Gamma(f,g) = |\mathsf{D}(f+g)|_w^2 - |\mathsf{D}f|_w^2 - |\mathsf{D}g|_w^2$$

as the 'scalar product of gradients'.

The bright side of $\mathsf{RCD}(K,\infty)$ spaces

IDEA: reinforce CD restricting to Riemannian-like metric-measure spaces only.

Ambrosio - Gigli - Savaré

Definition: $(X,\mathsf{d},\mathfrak{m})$ is $\mathsf{RCD}(K,\infty)$ if it is $\mathsf{CD}(K,\infty)$ and Ch is quadratic

Theorem (many people...)

Assume $(X,\mathsf{d},\mathfrak{m})$ has a quadratic Ch. TFAE:

$\mathsf{BE}(K,\infty)$: $\Gamma(\mathsf{P}_t f) \leq e^{-2Kt}\,\mathsf{P}_t\Gamma(f)$

Kuwada: $W_2(\mathsf{P}_t\mu,\mathsf{P}_t\nu) \leq e^{-Kt}\,W_2(\mu,\nu)$

$\mathsf{CD}(K,\infty)$: $\mathsf{Ent}_{\mathfrak{m}}(\mu_s) \leq (1-s)\mathsf{Ent}_{\mathfrak{m}}(\mu_0) + s\,\mathsf{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}s(1-s)\,W_2^2(\mu_0,\mu_1)$

EVI_K: $\frac{\mathrm{d}}{\mathrm{d}t}\frac{W_2^2(\mathsf{P}_t\mu,\nu)}{2} + \frac{K}{2}W_2^2(\mathsf{P}_t\mu,\nu) + \mathsf{Ent}(\mathsf{P}_t\mu) \leq \mathsf{Ent}(\nu)$

Here EVI_K stands for Evolution Variational Inequality and encodes the fact that the heat flow is the metric gradient flow of the entropy in the Wasserstein space.

A Carnot group $\mathbb{G}$ is a connected, simply connected, stratified Lie group with

$$\mathsf{Lie}(\mathbb{G}) = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \quad V_i = [V_1, V_{i-1}], \quad [V_1, V_\kappa] = \{0\}.$$

The horizontal directions $V_1 = \mathrm{span}\{X_1, \dots, X_m\}$, $m \in \mathbb{N}$, provide

$$\nabla_{\mathbb{G}} f = \textstyle\sum_{j=1}^m (X_j f) X_j \quad \text{and} \quad \Delta_{\mathbb{G}} = \textstyle\sum_{j=1}^m X_j^2.$$

One can identify $\mathbb{G} \sim (\mathbb{R}^n, \bullet)$ with Haar measure the Lebesgue measure $\mathscr{L}^n$.

We want to move only along V_1, so the Carnot-Carathéodory distance is

$$\mathsf{d}_{CC}(x,y) = \inf\left\{ \int_0^1 \|\dot\gamma_s\|_{\mathbb{G}}\, ds : \ \gamma_0 = x, \ \gamma_1 = y, \ \dot\gamma_t \in V_1 \right\}.$$

The space $(\mathbb{G}, \mathsf{d}_{cc}, \mathscr{L}^n)$ is Polish, geodesic and $\mathscr{L}^n(\mathsf{B}_{CC}(x,r)) = Cr^Q$, $Q \in \mathbb{N}$.

Theorem (Ambrosio - S., 2018)

The metric-measure space $(\mathbb{G}, \mathsf{d}_{CC}, \mathscr{L}^n)$ is NOT $\mathsf{CD}(K,\infty)$!

The case of the Heisenberg group $\mathbb{G} = \mathbb{H}^n$ was known since [Juillet, 2014].

The dark side of non-$\mathsf{CD}(K,\infty)$ spaces, part II: the $\mathbb{SU}(2)$ group

$\mathbb{SU}(2)$ = Lie group of 2×2 complex unitary matrices with determinant 1.

Lie algebra $\mathfrak{su}(2)$ = 2×2 complex unitary skew-Hermitian matrices with trace 0.

A basis of $\mathfrak{su}(2)$ is given by the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad i\in\mathbb{C},$$

satisfying the relations

$$\boxed{[X,Y]=2Z, \quad [Y,Z]=2X, \quad [Z,X]=2Y.}$$

Similarly as before, the horizontal generators X, Y provide $\boxed{\mathsf{d}_{\mathrm{CC}}}$ and

$$\boxed{\nabla_{\mathbb{SU}(2)} f = (Xf)X + (Yf)Y} \quad \text{and} \quad \boxed{\Delta_{\mathbb{SU}(2)} = X^2+Y^2.}$$

Using the cylindric coordinates (for $r\in[0,\frac{\pi}{2})$, $\vartheta\in[0,2\pi]$ and $\zeta\in[-\pi,\pi]$)

$$(r,\vartheta,z)\mapsto \exp(r\cos\vartheta\, X + r\sin\vartheta\, Y)\,\exp(\zeta\, Z) = \begin{pmatrix} e^{i\zeta}\cos r & e^{i(\vartheta-\zeta)}\sin r \\ -e^{-i(\vartheta-\zeta)}\sin r & e^{-i\zeta}\cos r \end{pmatrix},$$

the Haar measure $\mathfrak{m}\in\mathscr{P}(\mathbb{SU}(2))$ can be written as $\boxed{\mathrm{d}\mathfrak{m} = \frac{1}{4\pi^2}\,\sin(2r)\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\zeta.}$

The space $(\mathbb{SU}(2),\mathsf{d}_{\mathrm{CC}},\mathfrak{m})$ is Polish, geodesic and compact.

Why Carnot groups and $\mathbb{SU}(2)$ are interesting?

Theorem (Melcher, 2008)

Let $\mathbb{G}$ be a Carnot group. There exists $C_{\mathbb{G}} \geq 1$ such that $\Gamma^{\mathbb{G}}(\mathsf{P}_t f) \leq C_{\mathbb{G}}^2\, \mathsf{P}_t \Gamma^{\mathbb{G}}(f)$.

This is much weaker than usual BE, because we lose information at $t = 0$!
Remark: $C_{\mathbb{G}} = 1 \iff \mathbb{G}$ is commutative [Ambrosio-S., 2018].

Theorem (Baudoin - Bonnefont, 2008)

There exists $C_{\mathbb{SU}(2)} \geq \sqrt{2}$ such that $\Gamma^{\mathbb{SU}(2)}(\mathsf{P}_t f) \leq C_{\mathbb{SU}(2)}^2 e^{-4t}\, \mathsf{P}_t \Gamma^{\mathbb{SU}(2)}(f)$.

QUESTION: can we extend the equivalence

$$\text{BE} \iff \text{Kuwada} \iff \text{CD} \iff \text{EVI}$$

also to Carnot groups and $\mathbb{SU}(2)$?

NOTE: [Kuwada, 2009] already gives BE $\iff$ W_2-contraction.

We do not need smoothness: admissible metric-measure groups

Assume $(X, \mathsf{d}, \mathfrak{m})$ has Ch quadratic.

Definition (Admissible group)

$(X, \mathsf{d}, \mathfrak{m})$ is an admissible metric-measure group if:

- the metric space (X, d) is locally compact;
- the set X is a topological group, i.e. $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous;
- d is left-invariant, i.e. $\mathsf{d}(zx, zy) = \mathsf{d}(x, y)$ for all $x, y, z \in X$;
- $\mathfrak{m}$ is a left-invariant Haar measure, i.e. $\mathfrak{m}$ is a Radon measure such that $\mathfrak{m}(xE) = \mathfrak{m}(E)$ for all $x \in X$ and all Borel set $E \subset X$;
- X is unimodular, i.e. $\mathfrak{m}$ is also right-invariant.

REMARK: Carnot groups and $\mathbb{SU}(2)$ ARE admissible metric-measure groups.

Main result

Let $\mathsf{c}\colon [0,+\infty) \to (0,+\infty)$ be such that $\mathsf{c}, \mathsf{c}^{-1} \in \mathsf{L}^\infty([0,T])$ for all $T > 0$.

IDEA: c is a 'curvature function' and generalizes the usual $t \mapsto e^{-Kt}$.

Examples: $\mathsf{c}(t) \equiv C_{\mathbb{G}}$ for Carnot groups and $\mathsf{c}(t) = C_{\mathbb{SU}(2)} e^{-2t}$ for $\mathbb{SU}(2)$.

Define $\mathsf{R}(a,b) = \frac{1}{b-a} \int_a^b \mathsf{c}^{-2}(s)\,\mathsf{d}s$ for $0 \le a \le b$.

Theorem (S., 2020)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an admissible group + some technical hypotheses. TFAE:

BE_w: $\Gamma(\mathsf{P}_t f) \le \mathsf{c}^2(t)\, \mathsf{P}_t \Gamma(f)$

Kuwada: $W_2(\mathsf{P}_t\mu, \mathsf{P}_t\nu) \le \mathsf{c}(t)\, W_2(\mu,\nu)$

CD_w:
$$\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \le (1-s)\,\mathsf{Ent}_m(\mathsf{P}_t\mu_0) + s\,\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1) + \frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)} W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0, \mathsf{P}_t\mu_1)\right)$$
for $t \ge 0$ and $h > 0$, with $s \mapsto \mu_s$ a (1-speed) W_2-geodesic

EVI_w:
$$W_2^2(\mathsf{P}_{t_1}\mu_1, \mathsf{P}_{t_0}\mu_0) - \frac{1}{\mathsf{R}(t_0,t_1)} W_2^2(\mu_1,\mu_0) \le 2(t_1 - t_0)\Big(\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$$
for $0 \le t_0 \le t_1$

Comments

CD_w: $\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \le (1-s)\,\mathrm{Ent}_m(\mathsf{P}_t\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1) + \frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)}\,W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1)\right)$
for $t \ge 0$ and $h > 0$

EVI_w: $W_2^2(\mathsf{P}_{t_1}\mu_1,\mathsf{P}_{t_0}\mu_0) - \frac{W_2^2(\mu_1,\mu_0)}{\mathsf{R}(t_0,t_1)} \le 2(t_1-t_0)\Big(\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$
for $0 \le t_0 \le t_1$

1. The equivalence $\mathrm{BE}_w \iff$ Kuwada is known, see [Kuwada, 2009] and [Ambrosio - Gigli - Savaré, 2015], but we (re)do the proof because of some technical issues.
2. If $t = 0$ in CD_w, then $\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_h\mu_s) \le (1-s)\,\mathrm{Ent}_m(\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) + \frac{A(h)}{2}\,s(1-s)\,W_2^2(\mu_0,\mu_1)$ with $A(h) = \frac{\mathsf{R}(0,h)^{-1}-1}{h}$ for $h > 0$.
3. $\mathrm{CD}_w \implies$ Kuwada is easy: multiply by $h > 0$ and then send $h \to 0^+$.
4. $\mathrm{EVI}_w \implies \mathrm{CD}_w$ follows from a general argument, see [Daneri - Savaré, 2008].
5. We only need to prove $\mathrm{BE}_w \implies \mathrm{EVI}_w$. The proof is an adaptation of [Ambrosio - Gigli - Savaré, 2015] and [Erbar - Kuwada - Sturm, 2015].

Other comments and research directions

CD_w: $\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \le (1-s)\,\mathsf{Ent}_m(\mathsf{P}_t\mu_0) + s\,\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1)$
$$+\frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)}\,W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1)\right)$$
for $t \ge 0$ and $h > 0$

EVI_w: $W_2^2(\mathsf{P}_{t_1}\mu_1,\mathsf{P}_{t_0}\mu_0) - \frac{W_2^2(\mu_1,\mu_0)}{\mathsf{R}(t_0,t_1)} \le 2(t_1-t_0)\Big(\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$
for $0 \le t_0 \le t_1$

6. We need the group structure of X to exploit the de-singularization property of the convolution: $\varrho \star \mu \ll \mathfrak{m}$. Can we avoid this assumption? Example: metric graphs.

Note: $\mathsf{BE}_w \implies \mathsf{P}_t\mu \ll \mathfrak{m}$, but the W_2-metric velocity of $s \mapsto \mu_s^t = \mathsf{P}_t\mu_s$ cannot be related to the one of $s \mapsto \mu_s$ if $\mathsf{c}(0+) > 1$. Examples: Carnot groups and $\mathbb{SU}(2)$!

7. Consider a sub-Riemannian manifold $\mathbb{M}$ (possibly, without a group structure). Is there a BE_w inequality also encoding information about the dimension of $\mathbb{M}$?

8. $\mathsf{RCD}(K,\infty)$ and EVI_K imply several nice properties about $(X,\mathsf{d},\mathfrak{m})$ (MCP, gradient flows, m-GH stability,...). What can we deduce from RCD_w and EVI_w?

9. W_2-contractions are also known for Markovian diffusion semigroup associated to $L = \Delta + Z$ with $Z \in C^1$ on $(\mathbb{M}, \mathsf{g})$. Can we extend the result to this case?

THANK YOU FOR YOUR ATTENTION!

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