Bakry-Émery curvature condition and entropic inequalities on metric-measure groups

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Warm-up in $\mathbb{R}^N$

In $\mathbb{R}^N$ the solution of the heat equation

$$\begin{cases} \partial_t f_t = \Delta f_t & \text{on } \mathbb{R}^N \times (0, +\infty) \\ f_0 = f & \text{on } \mathbb{R}^N \end{cases}$$

is given by convolution as $\mathsf{P}_t f = \mathsf{p}_t * f$, where

$$\mathsf{p}_t(x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^N, \ t > 0,$$

is the heat kernel.

Hence we have $\nabla \mathsf{P}_t f = \mathsf{p}_t * (\nabla f) = \mathsf{P}_t \nabla f$, so that

$$\Gamma(\mathsf{P}_t f) = |\nabla \mathsf{P}_t f|^2 = |\mathsf{P}_t \nabla f|^2 \leq \mathsf{P}_t(|\nabla f|^2) = \mathsf{P}_t \Gamma(f)$$

by Jensen's inequality, since p_t is a probability measure. Thus

$$\boxed{\Gamma(\mathsf{P}_t f) \leq \mathsf{P}_t \Gamma(f)}$$

for all $t > 0$ and f sufficiently regular.

What happens in a Riemannian manifold? [1/2]

Let $(\mathbb{M}, \mathsf{g})$ be a smooth Riemannian manifold with Laplace-Beltrami operator Δ.

The heat flow $f_t = \mathsf{P}_t f$ starting from a datum f is associated to $\partial_t - \Delta$ as before.

Define $\varphi(s) = \mathsf{P}_s\Gamma(\mathsf{P}_{t-s}f)$ for $s \in [0,t]$ and $t > 0$, so that

$$\mathsf{P}_t\Gamma(f) - \Gamma(\mathsf{P}_t f) = \varphi(t) - \varphi(0) = \int_0^t \varphi'(s)\,ds.$$

One can check that

$$\varphi'(s) = 2\mathsf{P}_s\Gamma_2(\mathsf{P}_{t-s}f)$$

where

$$\Gamma(f,g) = \langle \nabla f, \nabla g\rangle_{\mathsf{g}}, \quad \Gamma_2(f,g) = \frac{1}{2}\big(\Delta\Gamma(f,g) - \Gamma(\Delta f, g) - \Gamma(f, \Delta g)\big).$$

The geometric meaning of Γ_2 is

$$\boxed{\Gamma_2(f) = ||\mathrm{Hess} f||_2^2 + \mathrm{Ric}(\nabla f, \nabla f),}$$

where $\mathrm{Ric}(\cdot,\cdot)$ is the Ricci tensor on $(\mathbb{M}, \mathsf{g})$.

What happens in a Riemannian manifold? [2/2]

Let us assume that, for some $K \in \mathbb{R}$,

$$\boxed{\mathrm{Ric}(v, v) \geq K|v|_{\mathrm{g}}^2,}$$

so that

$$\Gamma_2(f) = ||\mathrm{Hess} f||_2^2 + \mathrm{Ric}(\nabla f, \nabla f) \geq K\Gamma(f).$$

Consequently $\varphi'(s) = 2\mathsf{P}_s\Gamma_2(\mathsf{P}_{t-s}f) \geq 2K\mathsf{P}_s\Gamma(\mathsf{P}_{t-s}f) = 2K\varphi(s)$ and thus, by Grönwall inequality,

$$\boxed{\mathrm{Ric} \geq K \implies \Gamma(\mathsf{P}_t f) \leq e^{-2Kt}\,\mathsf{P}_t\Gamma(f)}$$

the Bakry-Émery-Ledoux pointwise gradient estimate for the heat flow.

If $\mathbb{M} = \mathbb{R}^N$, then $K = 0$ and we recover the Euclidean case.

To consider $N = \dim \mathbb{M}$ observe that $\boxed{||\mathrm{Hess} f||_2^2 \geq \frac{1}{N}(\Delta f)^2}$ [Wang, 2011].

Surprisingly, we have an equivalence:

$$\boxed{\mathsf{CD}(K, N) : \mathrm{Ric} \geq K,\ \dim \mathbb{M} \leq N \iff \Gamma_2(f) \geq \frac{1}{N}(\Delta f)^2 + K\,\Gamma(f),}$$

the Bakry-Émery curvature-dimension inequality (we will consider $N = \infty$ only).

Another equivalence via Wasserstein distance

We see $(\mathbb{M}, \mathsf{g})$ as a metric space (X, d) with $X = \mathbb{M}$ and $\mathsf{d} = \mathsf{d}_{\mathsf{g}}$.

Theorem (von Renesse - Sturm, 2005)

$$\mathrm{Ric} \geq K \iff W_2(\mathsf{P}_t\mu, \mathsf{P}_t\nu) \leq e^{-Kt} W_2(\mu, \nu) \text{ for all } \mu, \nu \in \mathscr{P}_2(\mathbb{M})$$

Here $(\mathscr{P}_2(X), W_2)$ is the Wasserstein metric space, where

$$\mathscr{P}_2(X) = \left\{ \mu \in \mathscr{P}(X) : \int_X \mathsf{d}(x, x_0)^2 \, \mathrm{d}\mu(x) < +\infty, \; x_0 \in X \right\}$$

and

$$W_2^2(\mu, \nu) = \inf \left\{ \int_{X \times X} \mathsf{d}^2(x, y) \, \mathrm{d}\pi : \pi(x, y) \in \mathrm{Plan}(\mu, \nu) \right\},$$

with

$$\mathrm{Plan}(\mu, \nu) = \{ \pi \in \mathscr{P}(X \times X) : (p_1)_\# \pi = \mu, \; (p_2)_\# \pi = \nu \}.$$

Important fact:

(X, d) Polish (geodesic) $\implies$ $(\mathscr{P}_2(X), W_2)$ Polish (geodesic).

Another equivalence via Boltzmann entropy

As before, we see $(X, \mathsf{d}, \mathfrak{m}) = (\mathbb{M}, \mathsf{g}, \mathrm{Vol}_{\mathsf{g}})$ as a metric-measure space.

Theorem (von Renesse - Sturm, 2005)

$$\mathrm{Ric} \geq K \iff \mathrm{Ent}_{\mathfrak{m}}(\mu_s) \leq (1-s)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2} s(1-s)\, W_2^2(\mu_0, \mu_1)$$

where $s \mapsto \mu_s$ is any (1-speed) W_2-geodesic joining $\mu_0, \mu_1 \in \mathrm{Dom}(\mathrm{Ent}_{\mathfrak{m}})$.

Here $\mathrm{Ent}_{\mathfrak{m}} : \mathscr{P}_2(X) \to (-\infty, +\infty]$ is the (Boltzmann) entropy

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) = \int_X \varrho \log \varrho \, \mathrm{d}\mathfrak{m}$$

for $\mu = \varrho\mathfrak{m} \in \mathscr{P}_2(X)$, with $\mathrm{Ent}_{\mathfrak{m}}(\mu) = +\infty$ if $\mu \not\ll \mathfrak{m}$.

NOTE: we want $\mathrm{Ent}(\mu) > -\infty$ for all $\mu \in \mathscr{P}_2(X)$, but this is OK whenever

$$\exists x_0 \in X \quad \exists A, B > 0 \quad : \quad \mathfrak{m}\,(B_r(x_0)) \leq A \exp\left(B\, r^2\right) \qquad \text{(exp.ball)}$$

Bishop volume comparison: $(X, \mathsf{d}, \mathfrak{m}) = (\mathbb{M}, \mathsf{g}, \mathrm{Vol}_{\mathsf{g}})$ with $\mathrm{Ric} \geq K \implies$ (exp.ball).

To be or not to be... smooth: the birth of $\mathsf{CD}(K, \infty)$ spaces

On a smooth Riemannian manifold $(\mathbb{M}, \mathsf{g})$ we know that

(1) Ric $\geq K$

(2) $\Gamma(\mathsf{P}_t) \leq e^{-2Kt}\,\mathsf{P}_t\Gamma$

(3) $W_2(\mathsf{P}_t, \mathsf{P}_t) \leq e^{-Kt}\,W_2$

(4) $\mathsf{Ent}_{\mathfrak{m}}$ is W_2-geodesic K-convex

are equivalent, but (4) only need d and $\mathfrak{m}$, not the smoothness of $(\mathbb{M}, \mathsf{g})$, hence making sense in metric-measure spaces.

Lott - Villani, Sturm

Definition: $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{CD}(K, \infty)$ space if $\mathsf{Ent}_{\mathfrak{m}}$ is W_2-geodesic K-convex

Natural questions:

What about (2) and (3)?

Can the heat flow be defined in a metric-measure space?

Some like it hot... and non-smooth

In a metric-measure space $(X, \mathsf{d}, \mathfrak{m})$, the Cheeger energy is

$$\mathsf{Ch}(f) = \inf \left\{ \liminf_n \int_X |\mathrm{D} f_n|^2 \, \mathrm{d}\mathfrak{m} : f_n \to f \text{ in } \mathsf{L}^2(X, \mathfrak{m}), \ f_n \in \mathrm{Lip}(X) \right\}$$

where $|\mathrm{D} f|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(x, y)}$ stands for the slope of $f : X \to \mathbb{R}$.

Ch is convex, l.s.c. and its domain $\mathsf{W}^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is dense in $L^2(X, \mathfrak{m})$

We can define the heat flow as the (Hilbertian) gradient flow of Ch in $L^2(X, \mathfrak{m})$:

$$\mathsf{P}_t f \xrightarrow[t \to 0^+]{} f \text{ in } \mathsf{L}^2(X, \mathfrak{m}) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{P}_t f \in -\partial^- \mathsf{Ch}(\mathsf{P}_t f) \text{ for a.e. } t > 0.$$

The Laplacian $-\Delta_{\mathsf{d},\mathfrak{m}} f \in \partial^- \mathsf{Ch}(f)$ is the element of minimal $\mathsf{L}^2(X, \mathfrak{m})$-norm.

CAUTION: $\mathsf{W}^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $\| \cdot \|_{\mathsf{W}^{1,2}} = \sqrt{\| \cdot \|_{\mathsf{L}^2}^2 + \mathsf{Ch}(\cdot)}$ may be NOT Hilbert!

Example: consider $(\mathbb{R}^n, \| \cdot \|_p, \mathscr{L}^n)$ for $p \neq 2$.

Any $f \in \mathsf{W}^{1,2}(X, \mathsf{d}, \mathfrak{m})$ has a (unique) weak gradient

$$\boxed{|\mathsf{D}f|_w \in \mathsf{L}^2(X, \mathfrak{m})}$$

such that

$$\boxed{\mathsf{Ch}(f) = \frac{1}{2}\int_X |\mathsf{D}f|_w^2 \, \mathsf{d}\mathfrak{m}.}$$

The weak gradient $|\mathsf{D}f|_w$ behaves like the 'modulus of the gradient' and one can develop Calculus rules in a non-smooth setting:

- locality: $|\mathsf{D}f|_w = |\mathsf{D}g|_w$ $\mathfrak{m}$-a.e. on $\{f - g = c\}$;
- Leibniz rule: $|\mathsf{D}(fg)|_w \le |f|\,|\mathsf{D}g|_w + |\mathsf{D}f|_w\,|g|$;
- chain rule: $\varphi \in \mathrm{Lip}(\mathbb{R}) \implies |\mathsf{D}\varphi(f)|_w \le |\varphi'(f)|\,|\mathsf{D}f|_w$.

Quadratic Cheeger energy

We say that Ch is quadratic if it satisfies the parallelogram law

$$\boxed{\mathsf{Ch}(f+g)+\mathsf{Ch}(f-g)=2\mathsf{Ch}(f)+2\mathsf{Ch}(g).}$$

We assume that Ch is quadratic, so that

$$W^{1,2}(X,\mathsf{d},\mathfrak{m}) \text{ is Hilbert}, \quad \mathsf{P}_t \text{ is linear}, \quad \Gamma(f)=|\mathrm{D}f|_w^2 \text{ is quadratic}.$$

By polarization, we can define

$$\boxed{\Gamma(f,g)=|\mathrm{D}(f+g)|_w^2-|\mathrm{D}f|_w^2-|\mathrm{D}g|_w^2}$$

as the 'scalar product of gradients':

- Leibniz rule: $\Gamma(fg,h)=g\,\Gamma(f,h)+f\,\Gamma(g,h)$;
- chain rule: $\Gamma(\varphi(f),g)=\varphi'(f)\,\Gamma(f,g)$;
- integration-by-parts: $\int_X \Gamma(f,g)\,\mathrm{d}\mathfrak{m}=-\int_X g\,\Delta_{\mathsf{d},\mathfrak{m}}f\,\mathrm{d}\mathfrak{m}$;
- Laplacian chain rule: $\Delta_{\mathsf{d},\mathfrak{m}}(\varphi\circ f)=\varphi'(f)\,\Delta_{\mathsf{d},\mathfrak{m}}f+\varphi''(f)\,\Gamma(f)$.

The bright side of $\mathsf{RCD}(K,\infty)$ spaces

IDEA: reinforce **CD** restricting to Riemannian-like metric-measure spaces only.

Ambrosio - Gigli - Savaré

Definition: $(X, \mathsf{d}, \mathfrak{m})$ is $\mathsf{RCD}(K,\infty)$ if it is $\mathsf{CD}(K,\infty)$ and **Ch** is quadratic

Theorem (many people...)

Assume $(X, \mathsf{d}, \mathfrak{m})$ has a quadratic **Ch**. TFAE:

$\mathsf{BE}(K,\infty)$: $\Gamma(\mathsf{P}_t f) \le e^{-2Kt}\,\mathsf{P}_t\Gamma(f)$

Kuwada: $W_2(\mathsf{P}_t\mu, \mathsf{P}_t\nu) \le e^{-Kt}\,W_2(\mu,\nu)$

$\mathsf{CD}(K,\infty)$: $\mathsf{Ent}_{\mathfrak{m}}(\mu_s) \le (1-s)\mathsf{Ent}_{\mathfrak{m}}(\mu_0) + s\,\mathsf{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}s(1-s)\,W_2^2(\mu_0,\mu_1)$

EVI_K: $\frac{\mathrm{d}}{\mathrm{d}t}\frac{W_2^2(\mathsf{P}_t\mu,\nu)}{2} + \frac{K}{2}W_2^2(\mathsf{P}_t\mu,\nu) + \mathsf{Ent}(\mathsf{P}_t\mu) \le \mathsf{Ent}(\nu)$

Here EVI_K stands for Evolution Variational Inequality and encodes the fact that the heat flow is the metric gradient flow of the entropy in the Wasserstein space.

What happens in the Heisenberg group? [1/2]

On the manifold $\mathbb{R}^3$ consider the non-commutative group operation

$$p \bullet q = (x, y, z) \bullet (x', y', z') = \left(x + x', y + y', z + z' + \tfrac{1}{2}(xy' - yx')\right).$$

The resulting Lie group $(\mathbb{R}^3, \bullet) \equiv \mathbb{H}^1$ is the (first) Heisenberg group.

There is a family of dilations: $\delta_\lambda(p) = (\lambda x, \lambda y, \lambda^2 z)$ for $\lambda > 0$.

The Haar measure is the Lebesgue measure $\mathscr{L}^3 = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

The tangent space is spanned by

$$X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z, \quad Z = [X, Y] = \partial_z.$$

We want to move only along the horizontal generators X, Y, so we define

$$\mathsf{d}_{CC}(p, q) = \inf\left\{ \int_0^1 \|\dot\gamma_s\|_{\mathbb{H}^1}\,\mathrm{d}s : \gamma_0 = p,\ \gamma_1 = q,\ \dot\gamma_s \in \mathrm{span}\{X_{\gamma_s}, Y_{\gamma_s}\}\right\}.$$

The function d_{CC} is the Carnot-Carathéodory (CC) distance [Chow - Rashevskii].

What happens in the Heisenberg group? [2/2]

The (sub-)Laplacian in $\mathbb{H}^1$ is

$$\boxed{\Delta_{\mathbb{H}^1} = X^2 + Y^2,}$$

which is only hypoelliptic: the heat kernel p_t of $\partial_t - \Delta_{\mathbb{H}^1}$ is smooth [Hörmander].

In $\mathbb{H}^1$ the solution of the (sub-elliptic) heat equation

$$\begin{cases} \partial_t f_t = \Delta_{\mathbb{H}^1} f_t & \text{on } \mathbb{R}^3 \times (0, +\infty) \\ f_0 = f & \text{on } \mathbb{R}^3 \end{cases}$$

is thus given by group convolution as

$$\boxed{\mathsf{P}_t f(p) = \mathsf{p}_t \star f(p) = \int_{\mathbb{R}^3} \mathsf{p}_t(q^{-1}p)\, f(q)\, \mathrm{d}q = \int_{\mathbb{R}^3} \mathsf{p}_t(q)\, f(pq^{-1})\, \mathrm{d}q.}$$

The horizontal gradient $\boxed{\nabla_{\mathbb{H}^1} = (X, Y)}$ is only left-invariant, so we are in troubles:

$$\nabla_{\mathbb{H}^1}(\mathsf{P}_t f) = \nabla_{\mathbb{H}^1}(\mathsf{p}_t \star f) = (\nabla_{\mathbb{H}^1} \mathsf{p}_t) \star f \neq \mathsf{p}_t \star (\nabla_{\mathbb{H}^1} f) = \mathsf{P}_t(\nabla_{\mathbb{H}^1} f).$$

Theorem (Juillet, 2009)

The metric-measure space $(\mathbb{H}^1, \mathsf{d}_{CC}, \mathscr{L}^3)$ is NOT $\mathsf{CD}(K, \infty)$!

A Carnot group $\mathbb{G}$ is a connected, simply connected, stratified Lie group with

$$\mathsf{Lie}(\mathbb{G}) = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \quad V_i = [V_1, V_{i-1}], \quad [V_1, V_\kappa] = \{0\}.$$

The horizontal directions $V_1 = \mathrm{span}\{X_1, \dots, X_m\}$, $m \in \mathbb{N}$, provide

$$\nabla_{\mathbb{G}} f = \sum_{j=1}^m (X_j f) X_j \quad \text{and} \quad \Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2.$$

One can identify $\mathbb{G} \sim (\mathbb{R}^n, \bullet)$ with Haar measure the Lebesgue measure $\mathscr{L}^n$.

We want to move only along V_1, so the Carnot-Carathéodory distance is

$$\mathsf{d}_{CC}(x,y) = \inf\left\{ \int_0^1 \|\dot\gamma_s\|_{\mathbb{G}}\, ds : \ \gamma_0 = x,\ \gamma_1 = y,\ \dot\gamma_t \in V_1 \right\}.$$

The space $(\mathbb{G}, \mathsf{d}_{cc}, \mathscr{L}^n)$ is Polish, geodesic and $\mathscr{L}^n(\mathsf{B}_{CC}(x,r)) = Cr^Q$, $Q \in \mathbb{N}$.

Example: for $\mathbb{H}^1$ it is $\kappa = 2$, $V_1 = \mathrm{span}\{X, Y\}$, $V_2 = \mathrm{span}\{Z\}$, $Q = 4$.

Theorem (Ambrosio - S., 2018)

The metric-measure space $(\mathbb{G}, \mathsf{d}_{CC}, \mathscr{L}^n)$ is NOT $\mathsf{CD}(K,\infty)$!

$\mathbb{SU}(2)$ = Lie group of 2×2 complex unitary matrices with determinant 1.

Lie algebra $\mathfrak{su}(2)$ = 2×2 complex unitary skew-Hermitian matrices with trace 0.

A basis of $\mathfrak{su}(2)$ is given by the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad i \in \mathbb{C},$$

satisfying the relations

$$[X,Y] = 2Z, \quad [Y,Z] = 2X, \quad [Z,X] = 2Y.$$

Similarly as before, the horizontal generators X, Y provide d_{CC} and

$$\nabla_{\mathbb{SU}(2)} f = (Xf)X + (Yf)Y \quad \text{and} \quad \Delta_{\mathbb{SU}(2)} = X^2 + Y^2.$$

Using the cylindric coordinates (for $r \in [0, \frac{\pi}{2})$, $\vartheta \in [0, 2\pi]$ and $\zeta \in [-\pi, \pi]$)

$$(r, \vartheta, z) \mapsto \exp(r \cos\vartheta\, X + r \sin\vartheta\, Y)\, \exp(\zeta\, Z) = \begin{pmatrix} e^{i\zeta} \cos r & e^{i(\vartheta - \zeta)} \sin r \\ -e^{-i(\vartheta-\zeta)} \sin r & e^{-i\zeta} \cos r \end{pmatrix},$$

the Haar measure $\mathfrak{m} \in \mathscr{P}(\mathbb{SU}(2))$ can be written as $\mathrm{d}\mathfrak{m} = \frac{1}{4\pi^2}\, \sin(2r)\, \mathrm{d}r\, \mathrm{d}\vartheta\, \mathrm{d}\zeta.$

The space $(\mathbb{SU}(2), \mathsf{d}_{CC}, \mathfrak{m})$ is Polish, geodesic and compact.

Why Carnot groups and $\mathbb{SU}(2)$ are interesting?

Theorem (Driver - Melcher, 2005)

There exists $C_{\mathbb{H}^1} > 1$ such that $\Gamma^{\mathbb{H}^1}(\mathsf{P}_t f) \leq C^2_{\mathbb{H}^1} \mathsf{P}_t \Gamma^{\mathbb{H}^1}(f)$.

This is much weaker than usual BE, because we lose information at $t = 0$!

Theorem (Melcher, 2008)

Let $\mathbb{G}$ be a Carnot group. There exists $C_{\mathbb{G}} \geq 1$ such that $\Gamma^{\mathbb{G}}(\mathsf{P}_t f) \leq C^2_{\mathbb{G}} \mathsf{P}_t \Gamma^{\mathbb{G}}(f)$.

Remark: $C_{\mathbb{G}} = 1 \iff \mathbb{G}$ is commutative [Ambrosio-S., 2018].

Theorem (Baudoin - Bonnefont, 2008)

There exists $C_{\mathbb{SU}(2)} \geq \sqrt{2}$ such that $\Gamma^{\mathbb{SU}(2)}(\mathsf{P}_t f) \leq C^2_{\mathbb{SU}(2)} e^{-4t} \mathsf{P}_t \Gamma^{\mathbb{SU}(2)}(f)$.

QUESTION: can we extend the equivalence

$$\text{BE} \iff \text{Kuwada} \iff \text{CD} \iff \text{EVI}$$

also to Carnot groups and $\mathbb{SU}(2)$?

NOTE: [Kuwada, 2009] already gives BE $\iff$ W_2-contraction.

We do not need smoothness: admissible metric-measure groups

Assume $(X, \mathsf{d}, \mathfrak{m})$ has Ch quadratic.

Definition (Admissible group)

$(X, \mathsf{d}, \mathfrak{m})$ is an admissible metric-measure group if:

- the metric space (X, d) is locally compact;
- the set X is a topological group, i.e. $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous;
- d is left-invariant, i.e. $\mathsf{d}(zx, zy) = \mathsf{d}(x, y)$ for all $x, y, z \in X$;
- $\mathfrak{m}$ is a left-invariant Haar measure, i.e. $\mathfrak{m}$ is a Radon measure such that $\mathfrak{m}(xE) = \mathfrak{m}(E)$ for all $x \in X$ and all Borel set $E \subset X$;
- X is unimodular, i.e. $\mathfrak{m}$ is also right-invariant.

REMARK: Carnot groups and $\mathbb{SU}(2)$ ARE admissible metric-measure groups.

Main result

Let $\mathsf{c}\colon [0,+\infty) \to (0,+\infty)$ be such that $\mathsf{c}, \mathsf{c}^{-1} \in \mathsf{L}^\infty([0,T])$ for all $T > 0$.

IDEA: c is a 'curvature function' and generalizes the usual $t \mapsto e^{-Kt}$.

Examples: $\mathsf{c}(t) \equiv C_{\mathbb{G}}$ for Carnot groups and $\mathsf{c}(t) = C_{\mathbb{SU}(2)} e^{-2t}$ for $\mathbb{SU}(2)$.

Define $\mathsf{R}(a,b) = \frac{1}{b-a} \int_a^b \mathsf{c}^{-2}(s)\, \mathrm{d}s$ for $0 \le a \le b$.

Theorem (S., 2020)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an admissible group + some technical hypotheses. TFAE:

BE$_w$: $\Gamma(\mathsf{P}_t f) \le \mathsf{c}^2(t)\, \mathsf{P}_t \Gamma(f)$

Kuwada: $W_2(\mathsf{P}_t\mu, \mathsf{P}_t\nu) \le \mathsf{c}(t)\, W_2(\mu,\nu)$

CD$_w$:
$$\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \le (1-s)\,\mathrm{Ent}_{m}(\mathsf{P}_t\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1) + \frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)} W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0, \mathsf{P}_t\mu_1)\right)$$

for $t \ge 0$ and $h > 0$, with $s \mapsto \mu_s$ a (1-speed) W_2-geodesic

EVI$_w$:
$$W_2^2(\mathsf{P}_{t_1}\mu_1, \mathsf{P}_{t_0}\mu_0) - \frac{1}{\mathsf{R}(t_0,t_1)} W_2^2(\mu_1,\mu_0) \le 2(t_1 - t_0)\Big(\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$$
for $0 \le t_0 \le t_1$

Comments

$$\mathsf{CD}_w: \ \mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \le (1-s)\,\mathrm{Ent}_{m}(\mathsf{P}_t\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1) + \frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)}\,W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1)\right)$$

for $t \ge 0$ and $h > 0$

$$\mathsf{EVI}_w: \ W_2^2(\mathsf{P}_{t_1}\mu_1,\mathsf{P}_{t_0}\mu_0) - \frac{W_2^2(\mu_1,\mu_0)}{\mathsf{R}(t_0,t_1)} \le 2(t_1-t_0)\Big(\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$$

for $0 \le t_0 \le t_1$

1. The equivalence $\mathsf{BE}_w \iff$ Kuwada is known, see [Kuwada, 2009] and [Ambrosio - Gigli - Savaré, 2015], but we (re)do the proof because of some technical issues.
2. If $t = 0$ in CD_w, then $\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_h\mu_s) \le (1-s)\,\mathrm{Ent}_{m}(\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) + \frac{A(h)}{2}\,s(1-s)\,W_2^2(\mu_0,\mu_1)$ with $A(h) = \frac{\mathsf{R}(0,h)^{-1}-1}{h}$ for $h > 0$.
3. $\mathsf{CD}_w \implies$ Kuwada is easy: multiply by $h > 0$ and then send $h \to 0^+$.
4. $\mathsf{EVI}_w \implies \mathsf{CD}_w$ follows from a general argument, see [Daneri - Savaré, 2008].
5. We only need to prove $\mathsf{BE}_w \implies \mathsf{EVI}_w$. The proof is an adaptation of [Ambrosio - Gigli - Savaré, 2015] and [Erbar - Kuwada - Sturm, 2015].

Other comments and futurama

CD_w: $\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \le (1-s)\,\mathrm{Ent}_m(\mathsf{P}_t\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1)$
$+\frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)}W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1)\right)$
for $t \ge 0$ and $h > 0$

EVI_w: $W_2^2(\mathsf{P}_{t_1}\mu_1,\mathsf{P}_{t_0}\mu_0) - \frac{W_2^2(\mu_1,\mu_0)}{\mathsf{R}(t_0,t_1)} \le 2(t_1-t_0)\Big(\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$
for $0 \le t_0 \le t_1$

1. We need the group structure of X to exploit the de-singularization property of the convolution: $\varrho \star \mu \ll \mathfrak{m}$. Can we avoid this assumption? Example: metric graphs.

Note: $\mathsf{BE}_w \implies \mathsf{P}_t\mu \ll \mathfrak{m}$, but the W_2-metric velocity of $s \mapsto \mu_s^t = \mathsf{P}_t\mu_s$ cannot be related to the one of $s \mapsto \mu_s$ if $\mathsf{c}(0+) > 1$. Examples: Carnot groups and $\mathbb{SU}(2)$!

2. Consider a sub-Riemannian manifold $\mathbb{M}$ (possibly, without a group structure). Is there a BE_w inequality also encoding information about the dimension of $\mathbb{M}$?

3. $\mathsf{RCD}(K,\infty)$ and EVI_K imply several nice properties about $(X,\mathsf{d},\mathfrak{m})$ (MCP, gradient flows, m-GH stability,...). What can we deduce from RCD_w and EVI_w?

4. W_2-contractions are also known for Markovian diffusion semigroup associated to $L = \Delta + Z$ with $Z \in C^1$ on $(\mathbb{M},\mathsf{g})$. Can we extend the result to this case?

Proof of $\mathsf{BE}_w \implies \mathsf{EVI}_w$ [1/6]

Let $s \in [0,1]$ and assume $\boxed{s \mapsto \mu_s = f_s \mathfrak{m}}$ is joining $\mu_0, \mu_1 \in \mathscr{P}_2(X)$.

Define a new curve $s \mapsto \tilde{\mu}_s = \tilde{f}_s \mathfrak{m}$ as

$$\boxed{\tilde{\mu}_s = \mathsf{P}_{\eta(s)} \mu_{\vartheta(s)},} \quad \text{so that} \quad \boxed{\tilde{f}_s = \mathsf{P}_{\eta(s)} f_{\vartheta(s)},}$$

where $\eta \in \mathsf{C}^2([0,1];[0,+\infty))$ and $\vartheta \in \mathsf{C}^1([0,1];[0,1])$ with $\vartheta(0) = 0$ and $\vartheta(1) = 1$.

At least formally, we can compute

$$\boxed{\frac{\mathsf{d}}{\mathsf{d}s} \tilde{f}_s = \dot{\eta}(s) \, \Delta \mathsf{P}_{\eta(s)} f_{\vartheta(s)} + \dot{\vartheta}(s) \, \mathsf{P}_{\eta(s)} \dot{f}_{\vartheta(s)}}$$

for $s \in (0,1)$.

$$\boxed{\frac{\mathsf{d}}{\mathsf{d}s}\,\tilde f_s = \dot\eta(s)\,\Delta \mathsf{P}_{\eta(s)} f_{\vartheta(s)} + \dot\vartheta(s)\,\mathsf{P}_{\eta(s)} \dot f_{\vartheta(s)}}$$

On the one hand, integrating by parts, we have

$$\begin{aligned}\frac{\mathsf{d}}{\mathsf{d}s}\,\mathsf{Ent}_{\mathfrak{m}}(\tilde\mu_s) &= \frac{\mathsf{d}}{\mathsf{d}s}\int_X \tilde f_s \log \tilde f_s \,\mathsf{d}\mathfrak{m}\\ &= \int_X (1+\log \tilde f_s)\,\frac{\mathsf{d}}{\mathsf{d}s}\,\tilde f_s \,\mathsf{d}\mathfrak{m}\\ &= -\,\dot\eta(s)\int_X p'(\tilde f_s)\,\Gamma(\tilde f_s)\,\mathsf{d}\mathfrak{m} + \dot\vartheta(s)\int_X p(\tilde f_s)\,\mathsf{P}_{\eta(s)}\dot f_{\vartheta(s)}\,\mathsf{d}\mathfrak{m}\end{aligned}$$

for $s \in (0,1)$, where $\boxed{p(r) = 1+\log r}$ for all $r > 0$.

Since $p'(r) = r(p'(r))^2$, by the chain rule $\Gamma(\varphi(f)) = (\varphi'(f))^2\,\Gamma(f)$ we can write

$$\frac{\mathsf{d}}{\mathsf{d}s}\,\mathsf{Ent}_{\mathfrak{m}}(\tilde\mu_s) = -\dot\eta(s)\int_X \Gamma(g_s)\,\mathsf{d}\tilde\mu_s + \dot\vartheta(s)\int_X \dot f_{\vartheta(s)}\,\mathsf{P}_{\eta(s)} g_s\,\mathsf{d}\mathfrak{m}$$

for $s \in (0,1)$, where $\boxed{g_s = p(\tilde f_s)}$ for brevity.

On the other hand, by Kantorovich duality, we have

$$\frac{1}{2} W_2^2(\mu,\nu) = \sup\left\{ \int_X Q_1\varphi \,\mathrm{d}\mu - \int_X \varphi \,\mathrm{d}\nu : \varphi \in \mathrm{Lip}(X) \text{ with bounded support} \right\},$$

where

$$Q_s\varphi(x) = \inf_{y\in X} \varphi(y) + \frac{\mathsf{d}^2(y,x)}{2s},$$

for $x \in X$ and $s > 0$, is the Hopf-Lax infimum-convolution semigroup.

Note that $\varphi_s = Q_s\varphi$ solves the Hamilton-Jacobi equation $\partial_s\varphi_s + \frac{1}{2}|\mathrm{D}\varphi_s|^2 = 0.$
Again integrating by parts, we can compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s \tilde{f}_s \,\mathrm{d}\mathfrak{m} &= \int_X \partial_s\varphi_s \,\mathrm{d}\tilde{\mu}_s + \int_X \varphi_s \frac{\mathrm{d}}{\mathrm{d}s}\tilde{f}_s \,\mathrm{d}\mathfrak{m} \\ &= -\frac{1}{2}\int_X \Gamma(\varphi_s)\,\mathrm{d}\tilde{\mu}_s - \dot{\eta}(s)\int_X \Gamma(\varphi_s, \tilde{f}_s)\,\mathrm{d}\mathfrak{m} \\ &\quad + \dot{\vartheta}(s)\int_X \dot{f}_{\vartheta(s)}\, \mathsf{P}_{\eta(s)}\varphi_s \,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Combining the above inequalities, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s\,\tilde f_s\,\mathrm{d}\mathfrak{m} + \dot\eta(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mathsf{Ent}_{\mathfrak{m}}(\tilde\mu_s) \le -\frac{1}{2}\int_X \big(\Gamma(\varphi_s) + \dot\eta(s)^2\,\Gamma(g_s)\big)\,\mathrm{d}\tilde\mu_s$$
$$- \dot\eta(s)\int_X \Gamma(\varphi_s,\tilde f_s)\,\mathrm{d}\mathfrak{m} + \dot\vartheta(s)\int_X \dot f_{\vartheta(s)}\,\mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\,g_s)\,\mathrm{d}\mathfrak{m}$$

for $s\in(0,1)$, forgetting the term $-\frac{\dot\eta(s)^2}{2}\int_X \Gamma(g_s)\,\mathrm{d}\tilde\mu_s \le 0$.

Now by non-smooth Calculus (since $r\,p'(r)=1$)

$$\Gamma(\varphi_s + \dot\eta(s)\,g_s) = \Gamma(\varphi_s) + 2\,\dot\eta(s)\,\Gamma(\varphi_s,g_s) + \dot\eta(s)^2\,\Gamma(g_s),$$
$$\Gamma(\varphi_s,g_s) = \Gamma(\varphi_s,p(\tilde f_s)) = p'(\tilde f_s)\,\Gamma(\varphi_s,\tilde f_s),$$
$$\Gamma(\varphi_s,g_s)\,\tilde f_s = \tilde f_s\,p'(\tilde f_s)\,\Gamma(\varphi_s,\tilde f_s) = \Gamma(\varphi_s,\tilde f_s),$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s\,\tilde f_s\,\mathrm{d}\mathfrak{m} + \dot\eta(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mathsf{Ent}_{\mathfrak{m}}(\tilde\mu_s) \le -\frac{1}{2}\int_X \Gamma(\varphi_s + \dot\eta(s)\,g_s)\,\mathrm{d}\tilde\mu_s$$
$$+ \dot\vartheta(s)\int_X \dot f_{\vartheta(s)}\,\mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\,g_s)\,\mathrm{d}\mathfrak{m}.$$

Proof of $\mathsf{BE}_w \implies \mathsf{EVI}_w$ [5/6]

At this point, the crucial information we need on $s \mapsto \mu_s = f_s \mathfrak{m}$ is that [Lisini]

$$\int_X \dot f_s \, \psi \, \mathsf{d}\mathfrak{m} \le |\dot\mu_s| \left(\int_X \Gamma(\psi) \, \mathsf{d}\mu_s \right)^{\frac{1}{2}} \qquad \text{(Lisini)}$$

for all 'nice' functions ψ, where

$$|\dot\mu_s| = \lim_{h \to 0} \frac{W_2(\mu_{s+h}, \mu_s)}{h}$$

is the metric velocity of the curve $s \mapsto \mu_s$ with respect to the Wasserstein distance.

We hence may choose $\psi = \mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\, g_s)$ and estimate

$$\begin{aligned}
\dot\vartheta(s) \int_X \dot f_{\vartheta(s)} \, \mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\, g_s) \, \mathsf{d}\mathfrak{m} &= \int_X \left(\frac{\mathsf{d}}{\mathsf{d}s} f_{\vartheta(s)} \right) \mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\, g_s) \, \mathsf{d}\mathfrak{m} \\
&\overset{\text{(Lisini)}}{\le} |\dot\vartheta(s)| \, |\dot\mu_{\vartheta(s)}| \left(\int_X \Gamma(\mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\, g_s)) \, \mathsf{d}\mu_s \right)^{\frac{1}{2}} \\
&\overset{\text{(C-S)}}{\le} \frac{\mathsf{c}^2(\eta(s))}{2} \, \dot\vartheta(s)^2 \, |\dot\mu_{\vartheta(s)}|^2 + \frac{\mathsf{c}^{-2}(\eta(s))}{2} \int_X \Gamma(\mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\, g_s)) \, \mathsf{d}\mu_s \\
&\overset{(\mathsf{BE}_w)}{\le} \frac{\mathsf{c}^2(\eta(s))}{2} \, \dot\vartheta(s)^2 \, |\dot\mu_{\vartheta(s)}|^2 + \frac{1}{2} \int_X \Gamma(\varphi_s + \dot\eta(s)\, g_s) \, \mathsf{d}\tilde\mu_s .
\end{aligned}$$

Proof of $\mathsf{BE}_w \implies \mathsf{EVI}_w$ [6/6]

By combining the above inequalities, we conclude that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s\,\tilde f_s\,\mathrm{d}\mathfrak{m} + \dot\eta(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Ent}_{\mathfrak{m}}(\tilde\mu_s) \le \frac{\mathsf{c}^2(\eta(s))}{2}\,\dot\vartheta(s)^2\,|\dot\mu_{\vartheta(s)}|^2}$$

for $s \in (0,1)$.

Choose $\boxed{\dot\vartheta(s) = \mathsf{c}^{-2}(\eta(s))}$ and integrate, so that, by Kantorovich duality, we get

$$\frac{1}{2}\,W_2^2(\mathsf{P}_{\eta(1)}\mu_1, \mathsf{P}_{\eta(0)}\mu_0) - \frac{1}{2\,\mathsf{R}(\eta)}\,W_2^2(\mu_1,\mu_0) + \dot\eta(1)\,\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{\eta(1)}\mu_1)$$
$$\le \dot\eta(0)\,\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{\eta(0)}\mu_0) + \int_0^1 \ddot\eta(s)\,\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{\eta(s)}\mu_{\vartheta(s)})\,\mathrm{d}s,$$

where $\mathsf{R}(\eta) = \int_0^1 \mathsf{c}^{-2}(\eta(s))\,\mathrm{d}s$.

No information on $\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_\eta\,\mu_\vartheta) \implies$ choose $\boxed{\eta(s) = (1-s)t_0 + st_1} \implies \mathsf{EVI}_w$.

THANK YOU FOR YOUR ATTENTION!

G. Stefani, Generalized Bakry-Émery curvature condition and equivalent entropic inequalities in groups, J. Geom. Anal. 32 (2022), no. 4, 136. Preprint available at arXiv:2008.13731.

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