A DISTRIBUTIONAL APPROACH TO FRACTIONAL SOBOLEV SPACES AND FRACTIONAL VARIATION

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Project and collaborators

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The fractional derivative: an old story, many definitions

Around 1675 Newton and Leibniz discovered Calculus. The concept of fractional derivative first appeared in a letter of Leibniz written to De l'Hôpital in 1695!

Today there are many fractional derivatives. Three famous examples:

Leibniz-Lacroix (1819): $$\frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Riemann-Liouville (1832-1847): $${}^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha}\, d\tau$$

Caputo (1967): $${}^{C}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau.$$

Some observations on fractional derivatives:

- they work on functions of just one variable;
- constants can have non-zero fractional derivative;
- they may need differentiable functions!

Question: What about a fractional gradient? Can we just take $(D_{e_1}^\alpha, \ldots, D_{e_n}^\alpha)$?

Problem: the 'coordinate approach' does not ensure invariance by rotations!

A 'physical' approach: invariance properties

Silhavy proposed that a (physically) 'good' fractional derivative should satisfy:

- invariance with respect to translations and rotations;
- α-homogeneity for some $\alpha \in (0,1)$;
- mild continuity on smooth functions.

For $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, we consider

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(y-x)}{|y-x|^{n+\alpha+1}}\,dy \in \mathbb{R}^n,$$

$$\mathrm{div}^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy \in \mathbb{R}$$

whenever $x \in \mathbb{R}^n$.

Theorem (Silhavy, 2020)

∇^α and div^α are determined (up to mult. const.) by the three requirements above.

A glimpse of the literature

Appearance

1959 Horvath (earliest reference up to knowledge)

1961 Nikol'ski-Sobolev (implicitly mentioned)

Variants, motivated by non-local interactions

1971 Edelen-Laws, Edelen-Green-Laws for non-local thermodynamics and continuum mechanics

2011-13-15 Caffarelli-Vazquez, Caffarelli-Soria-Vazquez, Biler-Imbert-Karch for non-local porous medium equation

Current research

2015-18 Shieh-Spector for fractional PDE theory (systematic study of ∇^α)

2017-20 Schikorra-Spector-Van Schaftingen, Spector for optimal embeddings

2019 Spector for potential theory

2020 Silhavy for distributional approach (introducing div^α)

2020-21 Bellido-Cueto-Mora-Corral for polyconvexity

2021 Saadi-Lakhal-Slimani for semilinear fractional elliptic equations

2022 Kreisbeck-Schönberger for quasiconvexity

Links with fractional Laplacian, Riesz transform and duality

Fractional Laplacian: $-\operatorname{div}^{\beta}\nabla^{\alpha} = (-\Delta)^{\frac{\alpha+\beta}{2}}$, where

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y) - f(x)}{|y-x|^{n+\alpha}}\, dy.$$

Riesz potential: $\nabla^{\alpha} = \nabla I_{1-\alpha}$ and $\operatorname{div}^{\alpha} = \operatorname{div} I_{1-\alpha}$, where

$$I_{\alpha} f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Integrability: $f, \varphi \in \operatorname{Lip}_c \implies \nabla^{\alpha} f,\ \operatorname{div}^{\alpha}\varphi \in L^1 \cap L^{\infty}$.

Duality: the integration-by-parts formula

$$\int_{\mathbb{R}^n} f \operatorname{div}^{\alpha}\varphi\, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^{\alpha} f\, dx$$

holds for $f \in \operatorname{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$.

IDEA: we can use the integration-by-parts formula to get a distributional theory!

A new path to Bessel potential spaces

For $p \in [1,+\infty]$, we define the distributional fractional Sobolev space

$$S^{\alpha,p}(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n) : \exists \nabla^\alpha f \in L^p(\mathbb{R}^n;\mathbb{R}^n) \}.$$

Here $\nabla^\alpha f \in L^1_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ is the weak fractional gradient of $f \in L^p(\mathbb{R}^n)$, i.e.

$$\int_{\mathbb{R}^n} f \,\text{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n).$$

Theorem (Bruè-Calzi-Comi-S. 2020, Kreisbeck-Schönberger 2022)

If $p \in (1,+\infty)$, then $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$, where

$$L^{\alpha,p}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : (I-\Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n) \}$$

is the Bessel potential space.

Meaning: '$H = W$-type result', since $L^{\alpha,p}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{L^p} + \|\nabla^\alpha \cdot\|_{L^p}}$.

Application: parallel Sobolev theory (PDEs, functionals) for Bessel potential spaces.

From a new concept of fractional variation...

Given $p \in [1, +\infty]$, the fractional variation of $f \in L^p(\mathbb{R}^n)$ is

$$|D^\alpha f|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1 \right\}$$

and we set the (Banach) space of L^p functions with bounded fractional variation as

$$BV^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty \right\}.$$

Theorem (Comi-S. 2019, Comi-Spector-S. 2022)

Variation measure: $f \in BV^{\alpha,p}(\mathbb{R}^n) \iff \int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f.$

$\mathscr{H}$-dim: $f \in BV^{\alpha,p}(\mathbb{R}^n) \implies |D^\alpha f| \ll \begin{cases} \mathscr{H}^{n-1} & \text{for } p \in \left[1, \frac{n}{1-\alpha}\right), \\ \mathscr{H}^{n-\alpha-\frac{n}{p}} & \text{for } p \in \left[\frac{n}{1-\alpha}, +\infty\right]. \end{cases}$

Embedding: $BV^{\alpha,p}(\mathbb{R}^n) \subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ for $p \in \left[1, \frac{n}{n-\alpha}\right)$ and $n \ge 2$.

Compactness: $BV^{\alpha,p}(\mathbb{R}^n) \subset L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is compact for $p \in \left[1, \frac{n}{n-\alpha}\right)$.

...to a new concept of fractional perimeter and reduced boundary

Given any open set $\Omega \subset \mathbb{R}^n$, the fractional Caccioppoli α-perimeter of E inside Ω is

$$|D^\alpha \chi_E|(\Omega) = \operatorname{sup}\left\{ \int_E \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\}.$$

The fractional reduced boundary $\mathscr{F}^\alpha E$ (inside Ω) is the set of points

$$x \in \operatorname{supp}(D^\alpha \chi_E) \quad \text{such that} \quad \exists \lim_{r\to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} \in \mathbb{S}^{n-1},$$

and the fractional (inner unit) normal at $x \in \Omega \cap \mathscr{F}^\alpha E$ is

$$\nu_E^\alpha \colon \Omega \cap \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}, \qquad \nu_E^\alpha(x) := \lim_{r\to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))}.$$

Observation: $\mathscr{L}^n(\mathscr{F}^\alpha E \cap \Omega) > 0$ for 'regular' sets (with BV-type jumps).

#1: $E = (a,b) \subset \mathbb{R} \implies \mathscr{F}^\alpha E = \mathbb{R} \setminus \left\{\frac{a+b}{2}\right\}$.

#2: $E = B_r(x_0) \subset \mathbb{R}^n \implies \mathscr{F}^\alpha E = \mathbb{R}^n \setminus \{c\}$ with $\nu_E^\alpha = \nu_{B_r(c)}$.

#3: $E = H_\nu^+(x_0) = \{(x - x_0)\cdot \nu \ge 0\} \implies \mathscr{F}^\alpha E = \mathbb{R}^n$ with $\nu_E^\alpha = \nu$.

The fractional version of De Giorgi's Theorem: existence of blow-ups

The blow-ups at $x \in \mathbb{R}^n$ of a set $E \subset \mathbb{R}^n$ are the family

$$\mathrm{Tan}(E,x) = \left\{ \text{limit points of } \left(\tfrac{E-x}{r}\right)_{r>0} \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+ \right\}.$$

Theorem (Comi-S., 2019)

- $\mathrm{Tan}(E,x) \neq \emptyset$ for all $x \in \mathscr{F}^\alpha E$.
- $F \in \mathrm{Tan}(E,x) \implies \nu_F^\alpha(y) = \nu_E^\alpha(x)$ for $|D^\alpha \chi_F|$-a.e. $y \in \mathscr{F}^\alpha F$.

Open problem: how to characterize blow-ups?

In the BV setting one uses coarea formula, (local) Poincaré inequality...

Bad News Theorem (Comi-Stefani, 2019-22)

The coarea formula and the local chain rule do NOT hold!

$$|D^\alpha f| \neq \int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|\, dt \qquad \text{and} \qquad |D^\alpha \Phi(f)| \neq |\Phi'|\,|D^\alpha f|$$

Key tools: density estimates, integration-by-parts formulas, Hardy inequalities...

Asymptotics

Analogues of Bourgain-Brezis-Mironescu and Ambrosio-De Philippis-Martinazzi:

Asymptotics for $\alpha \to 1^-$ (Comi-S., 2019)

- $f \in W^{1,p}(\mathbb{R}^n) \implies \nabla^\alpha f \to \nabla f$ in L^p whenever $p \in [1, +\infty)$.
- $f \in BV(\mathbb{R}^n) \implies D^\alpha f \rightharpoonup Df,\ |D^\alpha f| \rightharpoonup |Df|,\ |D^\alpha f|(\mathbb{R}^n) \to |Df|(\mathbb{R}^n)$.
- Γ-limits of fractional Caccioppoli perimeter and variation to integer analogues.

Analogues of Maz'ya-Shaposhnikova:

Asymptotics for $\alpha \to 0^+$ (Bruè-Calzi-Comi-S., 2020)

- $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n) \implies \nabla^\alpha f \to Rf$ in L^p whenever $p \in (1, +\infty)$.
- $f \in H^1(\mathbb{R}^n) \cap \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n) \implies \nabla^\alpha f \to Rf$ in L^1 and in H^1.
- $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n) \implies \alpha \int_{\mathbb{R}^n} |\nabla^\alpha f(x)|\, dx \to c_n \left| \int_{\mathbb{R}^n} f(x)\, dx \right|$.

The Hardy space is $H^1 = \{f \in L^1 : Rf \in L^1\}$, with $R = \nabla^0$ the Riesz transform.

Key tools: energy formulas, (new) fractional interpolation inequalities, density,...

Open problems and research directions

About sets and perimeter

- Is there a set with $\mathscr{L}^n$-negligible fractional reduced boundary?
- Can blow-ups be characterized in a more precise way?
- Are balls isoperimetric sets for the fractional variation?
- What about minimal surfaces for the fractional variation (existence, regularity)?

About functions and variation

- Can the fractional variation be (lower) bounded with the Hausdorff measure?
- Do $BV^{\alpha,p}$ functions satisfy some local properties (approximate limits, jumps)?
- Can $BV^{\alpha,p}$ functions be defined on a general open set $\Omega \subset \mathbb{R}^n$?

Thank you for your attention!