An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces

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Euler equations, velocity form

The Euler equations for an incompressible inviscid 2-dimensional fluid are given by

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \nabla p = 0 & \text{in } (0, +\infty) \times \Omega, \\ \operatorname{div} v = 0 & \text{in } [0, +\infty) \times \Omega, \\ v \cdot \nu_\Omega = 0 & \text{on } [0, +\infty) \times \partial\Omega, \\ v|_{t=0} = v_0 & \text{on } \Omega. \end{cases}$$

Objects:

- Ω is a sufficiently smooth (possibly unbounded) open set or the flat torus $\mathbb{T}^2$;
- $v \colon [0, +\infty) \times \Omega \to \mathbb{R}^2$ is the velocity of the fluid;
- $p \colon [0, +\infty) \times \Omega \to \mathbb{R}$ is the (scalar) pressure;
- $\nu_\Omega \colon \partial\Omega \to \mathbb{R}^2$ is the inner unit normal to $\partial\Omega$.

Conditions:

- $\operatorname{div} v = 0$ is the incompressibility condition;
- $v \cdot \nu_\Omega = 0$ at the boundary is the no-flow (or slip) condition.

Note: either $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{T}^2 \Rightarrow$ no boundary condition is imposed.

Euler equations, vorticity form

The vorticity $\omega\colon [0,+\infty)\times\Omega\to\mathbb{R}$ of the fluid is

$$\omega = \operatorname{curl} v$$

and satisfies

$$\begin{cases} \partial_t\omega + \operatorname{div}(v\omega) = 0 & \text{in } (0,+\infty)\times\Omega, \\ v = K\omega & \text{in } [0,+\infty)\times\Omega, \\ \omega|_{t=0} = \omega_0 & \text{on } \Omega. \end{cases}$$

Biot-Savart law: The relation $\omega = Kv$ is the Biot-Savart law, i.e.

$$v(t,x) = K\omega(t,x) = \int_\Omega k(x,y)\,\omega(t,y)\,dy,$$

where $k\colon \Omega\times\Omega\to\mathbb{R}^2$ is a convolution kernel.

Example: If $\Omega=\mathbb{R}^2$, then $k(x,y)=k_2(x-y)$ with

$$k_2(x) = \frac{1}{2\pi}\,\frac{1}{|x|^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \frac{1}{2\pi}\,\frac{x^\perp}{|x|^2} \quad \text{for all } x\in\mathbb{R}^2,\ x\neq 0.$$

Literature: a quick review

Theory of strong solutions is classical (since Lichtenstein 1930).

Existence of weak solutions:

- Yudovich (1963) for $L^1 \cap L^\infty$ vorticity
- DiPerna-Majda (1987), Delort (1991), Majda (1993), Vecchi-Wu (1993), Evans-Müller (1994) for L^1 vorticity
- Serfati (1995), Vishik (1999), Taniuchi (2004) for non-decaying vorticity

Uniqueness of weak solutions:

- Yudovich (1963) for $L^1 \cap L^\infty$ vorticity
- Yudovich (1995) for unbounded vorticity with L^p-norm mildly growing
- Vishik (1999) for ∞-Besov vorticity

Philosophy: while existence follows the usual pattern

smoothing data $\to$ existence of smooth solutions $\to$ compactness,

uniqueness is hard, due to non-linearity of Euler equations.

Warning: uniqueness is NOT expected for vorticity in L^p with $p < +\infty$!

- Vishik (2018), Albritton-Bruè-Colombo-De Lellis-Giri-Janisch-Kwon (2021)
- Bressan-Murray (2020), Bressan-Shen (2021)
- Bruè-Colombo (2021)

Yudovich's well-posedness for $L^1 \cap L^\infty$

Recall the Euler 2D equations in vorticity form:

$$\begin{cases} \partial_t \omega + \mathrm{div}(v\omega) = 0 & \text{in } (0,+\infty) \times \Omega, \\ v = K\omega & \text{in } [0,+\infty) \times \Omega, \\ \omega|_{t=0} = \omega_0 & \text{on } \Omega. \end{cases} \tag{E}$$

Theorem (Yudovich 1963)

There is a unique weak solution (ω, v) of (E) such that

$$\omega \in L^\infty([0,+\infty); L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \qquad v \in L^\infty([0,+\infty); C_b(\mathbb{R}^2; \mathbb{R}^2))$$

starting from $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $v_0 = K\omega_0$. Actually, the velocity satisfies

$$|v(t,x) - v(t,y)| \lesssim |x-y| \cdot |\log|x-y|| \quad x, y \in \mathbb{R}^2, \; t \geq 0.$$

Proof:

- existence relies on compactness of smooth solutions;
- uniqueness follows from a clever energy method.

Yudovich's energy method

Assume (ω^1, v^1) and (ω^2, v^2) are two weak solutions with same initial datum.

Consider the relative energy

$$E(t) = \int_{\mathbb{R}^2} |v^1(t,x) - v^2(t,x)|^2 \, dx \quad \text{for } t \geq 0.$$

Since $v = K\omega$, then $\nabla v = (\nabla K)\omega$, with ∇K a Calderón-Zygmund operator. Hence

$$\|\nabla v\|_{L^p} \lesssim p \, \|\omega\|_{L^p} \leq Cp \quad \text{for all } p \gg 1$$

where $C > 0$ depends on $\|\omega\|_{L^1}$ and $\|\omega\|_{L^\infty}$ only.

Exploit the Euler equations in velocity form $\partial_t v + (v \cdot \nabla)v + \nabla p = 0$ to get

$$\frac{d}{dt} E(t) \leq Cp \, E(t)^{1-1/p} \quad \text{for } t \in [0,T],$$

where $T > 0$ has to be chosen.

By comparison with the maximal solution of the ODE, we get

$$E(t) \leq (Ct)^p \leq (CT)^p \quad \text{for } t \in [0,T],$$

so that $E(t) = 0$ for all $t \in [0,T]$ letting $p \to +\infty$, provided that $CT < 1$.

Yudovich's energy method revised

If $\|\omega\|_{L^p} \lesssim \Theta(p)$ for $p \gg 1$ for some growth function $\Theta \colon [1, +\infty) \to (0, +\infty)$, then

$$\|\nabla v\|_{L^p} \lesssim p \, \|\omega\|_{L^p} \lesssim p \, \Theta(p) \quad \text{for all } p \gg 1.$$

Re-doing the same computations, one finds

$$\frac{d}{dt} E(t) \lesssim E(t) \, \psi_\Theta \left(\frac{1}{E(t)} \right) \quad \text{for } t \in [0, T],$$

where

$$\psi_\Theta(r) = \begin{cases} \inf\left\{ \frac{1}{\varepsilon} \, \Theta\left(1/\varepsilon \right) \ : \ \varepsilon \in (0, 1/3) \right\} & \text{for } r \in [0, 1), \\ \inf\left\{ \frac{1}{\varepsilon} \, \Theta\left(1/\varepsilon \right) \, r^\varepsilon \ : \ \varepsilon \in (0, 1/3) \right\} & \text{for } r \in [1, +\infty). \end{cases}$$

To show $E(t) = 0$ for $t \in [0, T]$, we just need $z \mapsto z \, \psi_\Theta(1/z)$ to satisfy

$$\int_{0+} \frac{dr}{r \, \psi_\Theta(1/r)} = +\infty,$$

the well-known Osgood condition.

Yudovich's well-posedness for Y^Θ

We let

$$Y^\Theta(\Omega) = \left\{ f \in \bigcap_{p\in[1,+\infty)} L^p(\Omega) \;:\; \|f\|_{Y^\Theta(\Omega)} = \sup_{p\in[1,+\infty)} \frac{\|f\|_{L^p(\Omega)}}{\Theta(p)} < +\infty \right\}$$

be the Yudovich space on Ω associated to Θ.

Theorem (Yudovich 1995)

Assume Θ is such that

$$\int_{0+} \frac{dr}{r\,\psi_\Theta(1/r)} = +\infty.$$

There is a unique weak solution (ω, v) of (E) such that

$$\omega \in L^\infty([0,+\infty); Y^\Theta(\mathbb{R}^2)) \qquad v \in L^\infty([0,+\infty); C_b(\mathbb{R}^2;\mathbb{R}^2))$$

starting from $\omega_0 \in Y^\Theta(\mathbb{R}^2)$, $v_0 = K\omega_0$. Actually, the velocity satisfies

$$|v(t,x) - v(t,y)| \lesssim |x-y| \cdot \psi_\Theta(1/|x-y|^3) \quad x, y \in \mathbb{R}^2,\ t \geq 0.$$

Examples of Θ

Bad news:

- energy method needs sharp tools (Sobolev spaces, CZ theory)
- behavior of ψ_Θ and its dependence on Θ are quite implicit!

Example: letting $\log_m p = \underbrace{\log\log\ldots\log}_{m \text{ times}} p$, Yudovich proved that

$$\Theta_m(p) \approx \log p \, \log_2 p \cdots \log_m p \;\Rightarrow\; \psi_{\Theta_m}(r) \approx \log r \, \log_2 r \cdots \log_{m+1} r.$$

The Osgood condition is

- true for $\Theta(p) \approx \log p$ (actually, for any Θ_m with $m \geq 1$)
- false for $\Theta(p) \approx p$.

This means that Yudovich's result

- applies for vorticities with singularities of order $|\log|\log|x||$
- does not apply for vorticities with singularities of order $|\log|x||$ (e.g., BMO)

Question: is there a more explicit relation between Θ and the modulus of continuity?

Properties of the kernel: less is more

Recall the Biot-Savart law is given by (dropping time dependence)

$$v(x) = K\omega(x) = \int_\Omega k(x,y)\,\omega(y)\,dy.$$

The convolution kernel $k\colon \Omega \times \Omega \to \mathbb{R}^2$ satisfies

- decay: $|k(x,y)| \le \dfrac{C_1}{|x-y|}$ for all $x, y \in \Omega$, $x \neq y$;
- oscillation: $|k(x,z) - k(y,z)| \le C_2 \dfrac{|x-y|}{|x-z|\,|y-z|}$ for all $x, y, z \in \Omega$, $z \neq x, y$;

for some constants $C_1, C_2 > 0$.

From the relation $v = K\omega$, we also get

- incompressibility: $\operatorname{div}(K\omega) = 0$;
- no-flow: $(K\omega) \cdot \nu_\Omega = 0$ at the boundary.

IDEA: try to rely on the above 'metric' properties of k only!

A posteriori: we can even relax the incompressibility property to

- controlled compression: $\|\operatorname{div}(K\omega)\|_{L^\infty(\Omega)} \le C_3 \|\omega\|_{L^1(\Omega)}$

for some constant $C_3 > 0$.

Exploit decay and oscillation

Fix $x, y \in \Omega$ with $d = |x-y| < 1$. We can split

$$|K\omega(x) - K\omega(y)| \leq \int_\Omega |k(x,z) - k(y,z)|\,|\omega(z)|\,dz$$
$$= \left(\int_{\Omega\setminus B_2(x)} + \int_{\Omega\cap(B_2(x)\setminus B_{2d}(x))} + \int_{\Omega\cap B_{2d}(x)}\right) |k(x,z) - k(y,z)|\,|\omega(z)|\,dz.$$

We can estimate

$$\int_{\Omega\setminus B_2(x)} \cdots \overset{\text{oscillation}}{\lesssim} |x-y| \int_{\Omega\setminus B_2(x)} \frac{|\omega(z)|}{|x-z|\,|y-z|}\,dz \lesssim |x-y|\,\|\omega\|_{L^1(\Omega)}$$

$$\int_{\Omega\cap(B_2(x)\setminus B_{2d}(x))} \cdots \overset{\text{oscillation}}{\lesssim} \int_{\Omega\cap(B_2(x)\setminus B_{2d}(x))} \frac{|\omega(z)|}{|x-z|^2}\,dz$$

$$\int_{\Omega\cap B_{2d}(x)} \cdots \overset{\text{decay}}{\lesssim} \int_{\Omega\cap B_{2d}(x)} \frac{|\omega(z)|}{|x-z|}\,dz + \int_{\Omega\cap B_{3d}(y)} \frac{|\omega(z)|}{|y-z|}\,dz$$

We need to control

$$\alpha(d) = \sup_{x\in\Omega} \int_{\Omega\cap(B_2(x)\setminus B_{2d}(x))} \frac{|\omega(z)|}{|x-z|^2}\,dz \quad \text{and} \quad \beta(d) = \sup_{x\in\Omega} \int_{\Omega\cap B_{3d}(x)} \frac{|\omega(z)|}{|x-z|}\,dz$$

defined for $d \in (0,1]$. By Hölder's inequality, we have

$$\alpha(d) \lesssim \left(\sup_{x\in\Omega} \|\omega\|_{L^p(\Omega\cap B_2(x))}\right) \left(\int_{2d}^{2} r^{1-2p'}\,dr\right)^{1/p'}$$
$$\lesssim C\,\left(\frac{2^{2-2p'}}{2p'-2}\right)^{1/p'} \left(d^{2-2p'}-1\right)^{1/p'} \lesssim C\,p\,d^{-2/p}$$

and, similarly,

$$\beta(d) \lesssim \left(\sup_{x\in\Omega} \|\omega\|_{L^p(\Omega\cap B_3(x))}\right) \left(\int_{0}^{3d} r^{1-p'}\,dr\right)^{1/p'}$$
$$\lesssim C\,\left(\frac{3^{2-p'}}{2-p'}\right)^{1/p'} d^{(2-p')/p'} \lesssim C\,\frac{p}{p-2}\,d^{1-2/p},$$

where $C = \sup_{x\in\Omega} \|\omega\|_{L^p(\Omega\cap B_1(x))}$.

We let

$$L^p_{\mathrm{ul}}(\Omega) = \left\{ f \in L^p_{\mathrm{loc}}(\Omega) \ : \ \|f\|_{L^p_{\mathrm{ul}}(\Omega)} = \sup_{x\in\Omega} \|f\|_{L^p(\Omega\cap B_1(x))} < +\infty \right\}$$

be the uniformly-localized L^p space on Ω. Note that radius $= 1$ is not restrictive.

Theorem (Hölder continuity)

Let $p \in (2, +\infty)$. If $\omega \in L^1(\Omega) \cap L^p_{\mathrm{ul}}(\Omega)$, then $K\omega \in C_b^{0,1-2/p}(\Omega;\mathbb{R}^2)$ with

$$\|K\omega\|_{L^\infty(\Omega;\mathbb{R}^2)} \lesssim \max\left\{1, \tfrac{1}{p-2}\right\} \left(\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L^p_{\mathrm{ul}}(\Omega)}\right)$$

$$|K\omega(x) - K\omega(y)| \lesssim \max\left\{1, \tfrac{1}{p-2}\right\} \left(\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L^p_{\mathrm{ul}}(\Omega)}\right) p\, |x-y|^{1-2/p} \quad \forall x, y \in \Omega.$$

Remark: the result is not a surprise, since (for the Biot-Savart kernel)

CZ theory + Morrey's inequality $\Rightarrow$ Hölder continuity.

However, our proof is surprising elementary!

Regularity of the velocity 2/3

We let

$$Y^{\Theta}_{\mathrm{ul}}(\Omega) = \left\{ f \in \bigcap_{p\in[1,+\infty)} L^p_{\mathrm{ul}}(\Omega) \ : \ \|f\|_{Y^{\Theta}_{\mathrm{ul}}(\Omega)} = \sup_{p\in[1,+\infty)} \frac{\|f\|_{L^p_{\mathrm{ul}}(\Omega)}}{\Theta(p)} < +\infty \right\}$$

be the uniformly-localized Yudovich space on Ω associated to Θ.

If $\omega \in L^1(\Omega) \cap Y^{\Theta}_{\mathrm{ul}}(\Omega)$, then for all $p \geq 3$ we have

$$\begin{aligned} |K\omega(x) - K\omega(y)| &\lesssim \max\left\{1, \tfrac{1}{p-2}\right\} (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L^p_{\mathrm{ul}}(\Omega)})\, p\, |x-y|^{1-2/p} \\ &\lesssim (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y^{\Theta}_{\mathrm{ul}}(\Omega)})\, \Theta(p)\, p\, |x-y|^{1-2/p}. \end{aligned}$$

If $d = |x-y| \ll 1$, then we can take $p = |\log d| \gg 1$ and observe that

$$\Theta(p)\, p\, |x-y|^{1-2/p} = \Theta(|\log d|)\, |\log d|\, d^{1-\frac{2}{|\log d|}} \approx d\, |\log d|\, \Theta(|\log d|)$$

since $d^{-\frac{2}{|\log d|}} = \exp(\frac{2}{\log d} \cdot \log d) = e^2$.

Regularity of the velocity 3/3

We let the function $\varphi_\Theta \colon [0,+\infty) \to [0,+\infty)$ be such that $\varphi_\Theta(0)=0$ and

$$\varphi_\Theta(r) = \begin{cases} r\,(1-\log r)\,\Theta(1-\log r) & \text{for } r \in (0,e^{-2}] \\ e^{-2}\,3\,\Theta(3) & \text{for } r > e^{-2}. \end{cases}$$

We say that φ_Θ is the modulus of continuity associated to Θ and define

$$C_b^{0,\varphi_\Theta}(\Omega;\mathbb{R}^2) = \left\{ v \in L^\infty(\Omega;\mathbb{R}^2) \ : \ \sup_{x \neq y} \frac{|v(x)-v(y)|}{\varphi_\Theta(|x-y|)} < +\infty \right\}.$$

Corollary (φ_Θ-continuity)

If $\omega \in L^1(\Omega) \cap Y_{\mathrm{ul}}^\Theta(\Omega)$, then $K\omega \in C_b^{0,\varphi_\Theta}(\Omega;\mathbb{R}^2)$ with

$$\|K\omega\|_{L^\infty(\Omega;\mathbb{R}^2)} \lesssim \|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y_{\mathrm{ul}}^\Theta(\Omega)}$$

$$|K\omega(x) - K\omega(y)| \lesssim \left(\|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y_{\mathrm{ul}}^\Theta(\Omega)}\right) \varphi_\Theta(|x-y|) \quad \forall x,y \in \Omega.$$

Remark: we recover Yudovich's continuity modulus (e.g., Θ_m), with NO sharp tools!

Existence

By definition of φ_Θ, note that

$$\int_{0^+} \frac{dr}{\varphi_\Theta(r)} = \int^{+\infty} \frac{dp}{p\,\Theta(p)}.$$

Theorem (Existence)

Let $p \in (2, +\infty)$. For any $\omega_0 \in L^1(\Omega) \cap L^p_{\mathrm{ul}}(\Omega)$, there is a weak solution (ω, v) of (E) such that

$$\omega \in L^\infty_{\mathrm{loc}}([0,+\infty); L^1(\Omega) \cap L^p_{\mathrm{ul}}(\Omega)) \qquad v \in L^\infty_{\mathrm{loc}}([0,+\infty); C_b^{0,1-2/p}(\Omega;\mathbb{R}^2)).$$

Moreover, if $\omega_0 \in L^1(\Omega) \cap Y^\Theta_{\mathrm{ul}}(\Omega)$, then (ω, v) is such that

$$\omega \in L^\infty_{\mathrm{loc}}([0,+\infty); L^1(\Omega) \cap Y^\Theta_{\mathrm{ul}}(\Omega)) \qquad v \in L^\infty_{\mathrm{loc}}([0,+\infty); C_b^{0,\varphi_\Theta}(\Omega;\mathbb{R}^2))$$

and, provided that φ_Θ is Osgood, (ω, v) is Lagrangian.

ODE theory: φ_Θ Osgood $\Rightarrow$ there is a unique flow X such that $\frac{d}{dt}X(t,\cdot) = v(t,X)$.

Lagrangian: the solution is such that $\omega(t,\cdot) = X(t,\cdot)_\# \omega_0$ (push-forward).

Remark: our existence result

- gives modulus of continuity even for Θ not BMO-like (e.g., $\Theta(p) \approx p^\alpha$);
- does not rely on the specific structure of the Biot-Savart kernel.

Proof of existence 1/2

Warning: we cannot rely on the existence of smooth solutions!

Indeed, the kernel is general, so there are no equations in velocity form.

We have to follow a different strategy:

1) construct a solution in $L^1 \cap L^\infty$ via time-stepping argument;
2) construct a solution in $L^1 \cap L^p_{\mathrm{ul}}$ by truncating the initial data;
3) show that the construction preserves the $L^1 \cap Y^\Theta_{\mathrm{ul}}$-regularity.

To gain existence, we need a compactness criterion à la Aubin-Lions:

- the proof exploits the Dunford-Pettis, Lusin and Arzelà-Ascoli Theorems;
- we assume weak compactness, while usually one takes strong compactness.

Proof of existence 2/2

Theorem (Baby Aubin-Lions)

Let $T > 0$ and let $(f^n)_{n\in\mathbb{N}} \subset L^\infty([0,T]; L^1(\Omega))$ be a bounded sequence which is equi-integrable in space uniformly in time:

- $\sup_{n\in\mathbb{N}} \|f^n\|_{L^\infty([0,T];\, L^1(\Omega))} < +\infty$
- $\forall \varepsilon > 0\ \exists \delta > 0 : A \subset \Omega,\ |A| < \delta \Rightarrow \sup_{n\in\mathbb{N}} \|f^n\|_{L^\infty([0,T];\, L^1(A))} < \varepsilon$
- $\forall \varepsilon > 0\ \exists \Omega_\varepsilon \subset \Omega$ with $|\Omega_\varepsilon| < +\infty\ :\ \sup_{n\in\mathbb{N}} \|f^n\|_{L^\infty([0,T];\, L^1(\Omega\setminus\Omega_\varepsilon))} < \varepsilon$.

Assume that, for each $\varphi \in C_c^\infty(\Omega)$, the functions $F_n[\varphi] \colon [0,T] \to \mathbb{R}$, given by

$$F_n[\varphi](t) = \int_\Omega f^n(t,\cdot)\,\varphi\,dx, \quad t \in [0,T],$$

are uniformly equi-continuous on $[0,T]$.

Then there exist a subsequence $(f^{n_k})_{k\in\mathbb{N}}$ and $f \in L^\infty([0,T]; L^1(\Omega))$ such that

$$\lim_{k\to+\infty} \int_\Omega f^{n_k}(t,\cdot)\,\varphi\,dx = \int_\Omega f(t,\cdot)\,\varphi\,dx$$

for a.e. $t \in [0,T]$ and all $\varphi \in L^\infty(\Omega)$.

Uniqueness

Theorem (Uniqueness)

Let Θ be such that φ_Θ is concave and Osgood. There is at most one (Lagrangian) weak solution (ω, v) of (E) such that

$$\omega \in L^\infty_{\mathrm{loc}}([0,+\infty); L^1(\Omega) \cap Y^\Theta_{\mathrm{ul}}(\Omega)) \qquad v \in L^\infty_{\mathrm{loc}}([0,+\infty); C^{0,\varphi_\Theta}_b(\Omega;\mathbb{R}^2)),$$

starting from $\omega_0 \in L^1(\Omega) \cap Y^\Theta_{\mathrm{ul}}(\Omega)$, $v_0 = K\omega_0$.

Remark: our uniqueness result

- recovers (and actually improves) Yudovich's uniqueness theorem;
- is proved in a Lagrangian way, we do not use the energy method;
- does not rely on the specific structure of the Biot-Savart kernel.

Careful: Osgood velocity $\Rightarrow$ any weak solution is Lagrangian, but this is delicate!

- Ambrosio-Bernard (2008)
- Caravenna-Crippa (2021)
- Clop-Jylhä-Mateu-Orotobig (2019)

Assume (ω^1, v^1) and (ω^2, v^2) are two Lagrangian solutions with same initial datum.

We can thus write $\omega^i = X^i(t,\cdot)_{\#}\omega_0$ where X^i is the flow associated to v^i, $i = 1, 2$.

Fix $T > 0$ and consider $t \in [0, T]$. We start with the usual splitting

$$\begin{aligned} |X^1 - X^2| &\leq \int_0^t |v^1(s, X^1) - v^2(s, X^2)|\, ds \\ &\leq \int_0^t |v^1(s, X^1) - v^1(s, X^2)|\, ds + \int_0^t |v^1(s, X^2) - v^2(s, X^2)|\, ds. \end{aligned}$$

The first term is easy, we can use the φ_Θ-continuity and obtain

$$|v^1(s, X^1) - v^1(s, X^2)| \lesssim \varphi_\Theta(|X^1 - X^2|),$$

with implicit constant depending on $\|\omega^1\|_{L^\infty([0,T];L^1 \cap Y^\Theta_{\mathrm{ul}})}$.

The second term is delicate. We use $v = K\omega$ and the push-forward to get

$$\begin{aligned}|v^1(s,X^2) - v^2(s,X^2)| &= |(K\omega^1)(s,X^2) - (K\omega^2)(s,X^2)| \\ &= \left| \int_\Omega k(X^2,y)\,\omega^1(s,y)\,dy - \int_\Omega k(X^2,y)\,\omega^2(s,y)\,dy \right| \\ &= \left| \int_\Omega k(X^2,X^1(s,y))\,\omega_0(y)\,dy - \int_\Omega k(X^2,X^2(s,y))\,\omega_0(y)\,dy \right| \\ &\leq \int_\Omega |k(X^2,X^1(s,y)) - k(X^2,X^2(s,y))|\,|\omega_0(y)|\,dy.\end{aligned}$$

We combine the two estimates and obtain

$$\begin{aligned}|X^1 - X^2| \leq \int_0^t \varphi_\Theta(|X^1 - X^2|)\,dt \\ + \int_0^t \int_\Omega |k(X^2,X^1(s,y)) - k(X^2,X^2(s,y))|\,|\omega_0(y)|\,dy\,dt.\end{aligned}$$

Now choose the finite measure $\mu = \bar\omega\,\mathscr{L}^2$, with $\bar\omega = |\omega_0| + \eta$ and $0 < \eta \in L^1 \cap L^\infty$.

Proof of uniqueness 3/4

We integrate with respect to μ. By Tonelli Theorem, we can estimate

$$\begin{aligned}
&\int_\Omega \int_\Omega |k(X^2(s,x), X^1(s,y)) - k(X^2(s,x), X^2(s,y))|\, |\omega_0(y)|\, dy\, d\mu(x) \\
&= \int_\Omega |\omega_0(y)| \int_\Omega |k(X^2(s,x), X^1(s,y)) - k(X^2(s,x), X^2(s,y))|\, d\mu(x)\, dy \\
&= \int_\Omega |\omega_0(y)| \int_\Omega |k(x, X^1(s,y)) - k(x, X^2(s,y))|\, X^2(s,\cdot)_\# \bar{\omega}(x)\, dx\, dy \\
&\overset{(!)}{\lesssim} \int_\Omega |\omega_0(y)|\, \varphi_\Theta(|X^1(s,y) - X^2(s,y)|)\, dy \\
&\le \int_\Omega \varphi_\Theta(|X^1(s,y) - X^2(s,y)|)\, d\mu(y).
\end{aligned}$$

Inequality (!) follows from the same computations for the φ_Θ-continuity of velocity.

The implicit constant depends on $\|\bar{\omega}\|_{L^\infty([0,T];L^1 \cap Y^\Theta_{\mathrm{ul}})}$. But $\bar{\omega} = |\omega_0| + \eta$, so we can choose $\eta \in L^1 \cap L^\infty$ to let the constant depend on $\|\omega_0\|_{L^\infty([0,T];L^1 \cap Y^\Theta_{\mathrm{ul}})}$ only!

Proof of uniqueness 4/4

In conclusion, we get

$$\int_\Omega |X^1 - X^2|\, d\mu \lesssim \int_0^t \int_\Omega \varphi_\Theta(|X^1 - X^2|)\, d\mu\, dt.$$

But φ_Θ is concave and Osgood, so that

$$\int_\Omega \varphi_\Theta(|X^1 - X^2|)\, d\mu \overset{\text{Young}}{\leq} \varphi_\Theta\left(\int_\Omega |X^1 - X^2|\, d\mu\right)$$

and thus

$$\xi(t) \leq \int_0^t \varphi_\Theta(\xi(s))\, dt, \qquad \xi(s) = \int_\Omega |X^1(s,\cdot) - X^2(s,\cdot)|\, d\mu,$$

imply that $X^1 = X^2$ for all $t \in [0,T]$, which means $\omega^1 = \omega^2$ and so $v^1 = v^2$.

Futurama

Project 1: apply this elementary approach to Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 \\ E = K\varrho \\ \varrho = \int f \, dv, \end{cases}$$

in collaboration with G. Crippa, T. Dolmaire and C. Saffirio.

Project 2: remove L^1 assumption, dealing with weak solutions in Y^{Θ}_{ul} for suitable Θ, in collaboration with G. Ciampa and G. Crippa.

Other ideas: more general functional spaces? other equations?

Thank you for your attention!

G. Crippa and G. Stefani, An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces (2021), submitted, available at arXiv:2110.15648.

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