ON THE CONVEX COMPONENTS OF A SET

GIORGIO STEFANI

ABSTRACT. The convex components of a set (with non-empty interior) are not uniquely determined in general. In this talk, after having recalled some elementary (quantitative) monotonicity properties of the Euclidean perimeter, we study some lower bounds on the minimal number of convex components of an arbitrary set, showing the sharpness of our results with some explicit examples. This is a joint work in collaboration with F. Giannetti.

1. ARCHIMEDES AND THE (QUANTITATIVE) MONOTONICITY OF PERIMETER

The ambient space is $\mathbb{R}^n$ with $n \geq 2$. For all $s \geq 0$, we let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure (in particular, $\mathcal{H}^0$ is the counting measure).

Definition 1.1 (Body)**.** A *body* $E \subset \mathbb{R}^n$ is a compact set with non-empty interior.

If $E \subset \mathbb{R}^n$ is a k-dimensional convex body, with $1 \leq k \leq n$, we let ∂E be its boundary, which is a set of Hausdorff dimension $(k-1)$.

Definition 1.2 (Perimeter)**.** If $E \subset \mathbb{R}^n$ is a convex body, then $P(E) = \mathcal{H}^{n-1}(\partial E)$ denotes the *perimeter* of E.

Proposition 1.3 (Monotonicity)**.** *If $A \subset B \subset \mathbb{R}^n$ are convex bodies, then*

$$P(A) \leq P(B). \tag{1.1}$$

Inequality (1.1) is well known since the ancient Greek (Archimedes himself took it as a postulate in his work on the sphere and the cylinder, see [1, p. 36]) and can be proved in many different ways, for example by exploiting either the Cauchy formula for the area surface or the monotonicity property of mixed volumes, [2, §7], by using the Lipschitz property of the projection on a convex closed set, [3, Lemma 2.4], or finally by observing that the perimeter is decreased under intersection with half-spaces, [10, Exercise 15.13].

Theorem 1.4 (Quantitative monotonicity [4, 5, 9, 11])**.** *Let $n \geq 2$. If $A \subset B$ are two convex bodies in $\mathbb{R}^n$, then*

$$P(A) + \frac{\omega_{n-1} r^{n-2} h^2}{r + \sqrt{r^2 + h^2}} \leq P(B), \tag{1.2}$$

Date: November 18, 2021.

2020 *Mathematics Subject Classification.* Primary 52A20; Secondary 52A40.

Key words and phrases. Convex body, convex component, monotonicity of perimeter, Hausdorff distance.

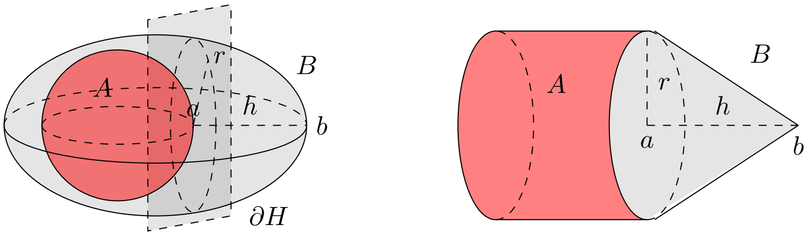

FIGURE 1. The setting of the estimate (1.2) (on the left) with an example of equality (on the right).

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ *denotes the volume of the unit ball in* $\mathbb{R}^n$, $h = h(A, B)$ *is the Hausdorff distance of* A *and* B *and*

$$r = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(B \cap \partial H)}{\omega_{n-1}}}, \qquad H = \{x \in \mathbb{R}^n : \langle b-a, x-a\rangle \le 0\},$$

with $a \in A$ *and* $b \in B$ *such that* $|a-b| = h(A, B)$, *see Figure 1 (left).*

The quantitative estimate (1.2) is sharp, in the sense that they hold as equalities in some cases, see Figure 1 (right).

2. Convex components

Let $n \ge 2$. Let $E \subset \mathbb{R}^n$ be a body and consider a decomposition of the form

$$E = \bigcup_{i=1}^{k} E_i, \tag{2.1}$$

where $k \in \mathbb{N}$ and $E_1, \dots, E_k$ are convex bodies, the *convex components* of E. In general, such a decomposition is obviously not unique, so we want to give a lower bound on the minimal number $k_{\min}(E) \in \mathbb{N}$ of the convex components of E.

We observe that:

- $k_{\min}(E) = 1$ if and only if E is a convex body;
- $k_{\min}(E) \ge c(E)$, where $c(E) \in \mathbb{N}$ is the number of connected components of E.

Therefore, without loss of generality, we can assume that E is connected.

The first lower bound on the minimal number of convex components was given in [9, Theorem 1.1]. Here and in the following, $\lceil x \rceil \in \mathbb{Z}$ denotes the upper integer part of $x \in \mathbb{R}$.

Theorem 2.1 (Lower bound)**.** *Let* $n \ge 2$. *If* $E \subset \mathbb{R}^n$ *is a body admitting a decomposition* (2.1), *then*

$$k_{\min}(E) \ge \left\lceil \frac{P(E)}{P(\mathrm{co}(E))} \right\rceil. \tag{2.2}$$

Taking advantage of the quantitative estimate (1.2), in [6] the authors were able to improve the lower bound (2.2) for $n = 2$.

Theorem 2.2 (Improved lower bound for $n = 2$)**.** *Let* $E \subset \mathbb{R}^2$ *be a body. Assume that there exist* $p \in \mathbb{N}$ *and* $\alpha \in (0, 1)$ *such that any decomposition of the form* (2.1) *admits* p

convex components $E_{i_1}, \dots, E_{i_p}$ *with*

$$h(E_{i_j}, \mathrm{co}(E)) \geq \alpha \operatorname{diam}(\mathrm{co}(E)) \tag{2.3}$$

for all $j = 1, \dots, p$. *Then*

$$k_{\min}(E) \geq \left\lceil \frac{P(E) + \frac{4\alpha^2 p}{1+\sqrt{1+4\alpha^2}} \operatorname{diam}(\mathrm{co}(E))}{P(\mathrm{co}(E))} \right\rceil. \tag{2.4}$$

Inequality (2.4) is sharp, in the sense that it holds as an equality in some cases. Moreover, it improves the previous lower bound (2.2) in the case $n = 2$. Indeed, in [6] the authors exhibit an example for which (2.2) gives a strict inequality while, on the contrary, (2.4) yields an equality.

Definition 2.3 (Maximal sectional radius)**.** Let $n \geq 2$ and let $E \subset \mathbb{R}^n$ be a body. Given a unitary direction $\nu \in \mathbb{S}^{n-1}$, we let

$$\rho_\nu(E) = \sup \left\{ \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(E \cap (t\nu + \partial H_\nu))}{\omega_{n-1}}} : t \in \mathbb{R} \right\}$$

be the *maximal sectional radius of* E *in the direction* ν, where $H_\nu = \{x \in \mathbb{R}^n : \langle x, \nu \rangle \geq 0\}$. Note that $\rho_{-\nu}(E) = \rho_\nu(E)$ for all $\nu \in \mathbb{S}^{n-1}$.

Our main result is the following, see [7, Theorem 3.3].

Theorem 2.4 (Improved lower bound for $n \geq 2$)**.** *Let* $n \geq 2$ *and let* $E \subset \mathbb{R}^n$ *be a body. Assume that there exist* $p \in \mathbb{N}$, $\alpha \in (0,1)$ *and* $\beta \in [0,1]$ *with the following properties. For every family* $E_1, \dots, E_k$, *with* $k \in \mathbb{N}$, *of convex bodies such that* $E = \bigcup_{i=1}^k E_i$, *we can find a subfamily of* p *convex bodies* $E_{i_1}, \dots, E_{i_p}$ *and a family of corresponding* p *closed half-spaces such that* $E_{i_j} \subset H_{i_j}$,

$$h(\mathrm{co}(E) \cap H_{i_j}, \mathrm{co}(E)) \geq \alpha \operatorname{diam}(\mathrm{co}(E)) \tag{2.5}$$

and

$$\mathcal{H}^{n-1}(\mathrm{co}(E) \cap \partial H_{i_j}) \geq \beta \omega_{n-1} \rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-1} \tag{2.6}$$

for all $j = 1, \dots, p$, *where* $\nu_{i_j} \in \mathbb{S}^{n-1}$ *is the inner unit normal of the half-space* H_{i_j}. *Then*

$$k_{\min}(E) \geq \left\lceil \frac{P(E) + \omega_{n-1}\alpha^2\beta^{n-2} \displaystyle\sum_{j=1}^{p} \frac{\rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-2} \operatorname{diam}(\mathrm{co}(E))^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + \alpha^2 \operatorname{diam}(\mathrm{co}(E))^2}}}{P(\mathrm{co}(E))} \right\rceil. \tag{2.7}$$

Inequality (2.7) improves the previous lower bound (2.2) since it holds as an equality in some cases for which (2.2) gives a strict inequality only. We will give some explicit examples in Section 4 below.

Note that the assumption (2.5) corresponds to (2.3), while the additional assumption (2.6) comes into play for $n \geq 3$ only.

In fact, if we take $n = 2$ in Theorem 2.4, then the inequality (2.7) becomes

$$k_{\min}(E) \geq \left\lceil \frac{P(E) + 2\alpha^2 \sum_{j=1}^{p} \frac{\operatorname{diam}(\operatorname{co}(E))^2}{\rho_{\nu_{i_j}}(\operatorname{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\operatorname{co}(E))^2 + \alpha^2 \operatorname{diam}(\operatorname{co}(E))^2}}}{P(\operatorname{co}(E))} \right\rceil \tag{2.8}$$

(as customary, we use the convention $0^0 = 1$) and the parameter $\beta \in [0,1]$ provided by (2.6) plays no role in the final estimate (2.8). Consequently, the additional assumption in (2.6) can be dropped and one just need to choose the closed half-plane $H_{i_j} \subset \mathbb{R}^2$ in such a way that

$$h(\operatorname{co}(E) \cap H_{i_j}, \operatorname{co}(E)) = h(E_{i_j}, \operatorname{co}(E)) \quad \text{for all } j = 1, \dots, p,$$

which is always possible by the definition of the Hausdorff distance and the convexity of each component E_{i_j}.

Concerning the higher dimensional case $n \geq 3$, a control like the one in (2.6) seems reasonable to be assumed. Indeed, as one may realize by looking at the inequality (2.2), the set $E \subset \mathbb{R}^n$ may have a convex component very lengthened in one specific direction $\nu \in \mathbb{S}^{n-1}$ which does not give a substantial contribution to the total perimeter of E but, nevertheless, that strongly affects the total perimeter of the convex hull $\operatorname{co}(E)$.

In addition, we observe that the effectiveness of the lower bound (2.2) drastically changes when passing from the planar case $n = 2$ to the non-planar case $n \geq 3$. Indeed, if $E \subset \mathbb{R}^2$ is a non-convex connected compact set admitting at least one decomposition like (2.1), then

$$P(\operatorname{co}(E)) < P(E),$$

correctly implying that $k_{\min}(E) \geq 2$. However, as we are going to show with some examples in Section 4 below, there are non-convex connected compact sets $E \subset \mathbb{R}^n$, with $n \geq 3$, such that

$$P(\operatorname{co}(E)) \geq P(E),$$

so that (2.2) only implies that $k_{\min}(E) \geq 1$. Nevertheless, the inequality (2.7) given by Theorem 2.4 allows us to recover the correct value of $k_{\min}(E)$ in these examples.

Last but not least, let us observe that, in the planar case $n = 2$, one can trivially bound

$$\rho_\nu(\operatorname{co}(E)) \leq \frac{\operatorname{diam}(\operatorname{co}(E))}{2} \quad \text{for all } \nu \in \mathbb{S}^1, \tag{2.9}$$

so that inequality (2.8) gives back

$$\begin{aligned} k_{\min}(E) &\geq \left\lceil \frac{P(E) + \frac{2p\alpha^2 \operatorname{diam}(\operatorname{co}(E))^2}{\frac{\operatorname{diam}(\operatorname{co}(E))}{2} + \sqrt{\frac{\operatorname{diam}(\operatorname{co}(E))^2}{4} + \alpha^2 \operatorname{diam}(\operatorname{co}(E))^2}}}{P(\operatorname{co}(E))} \right\rceil \\ &= \left\lceil \frac{P(E) + \frac{4\alpha^2 p}{1+\sqrt{1+4\alpha^2}} \operatorname{diam}(\operatorname{co}(E))}{P(\operatorname{co}(E))} \right\rceil, \end{aligned}$$

that is the estimate in (2.4). Actually, because of the fact that the upper bound (2.9) can be too rough in general, the inequality (2.7) given by our Theorem 2.4 is more precise than the one in (2.4), as we are going to show in Example 4.1 below.

3. Proof of Theorem 2.4

We recall that, if $A \subset B$ are two compact sets in $\mathbb{R}^n$, with $n \geq 2$, then the Hausdorff distance $h(A, B)$ between A and B can be written as

$$h(A, B) = \max_{b\in B} \operatorname{dist}(A, b) = \max_{b\in B} \min_{a\in A} |a - b|.$$

As above, given $\emptyset \neq A \subset B$ two convex bodies in $\mathbb{R}^n$, with $n \geq 2$, we denote by

$$\delta(B, A) := \mathcal{H}^{n-1}(\partial B) - \mathcal{H}^{n-1}(\partial A) \geq 0$$

the perimeter deficit between A and B.

Proof of Theorem 2.4. Since E is compact, its convex hull $\operatorname{co}(E)$ is compact too, see [8, Corollary 3.1] for example. As a consequence, $P(\operatorname{co}(E)) < +\infty$. Arguing as in [6], we can estimate

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial E) &\leq \mathcal{H}^{n-1}\left(\bigcup_{i=1}^{k} \partial E_i\right) \leq \sum_{i=1}^{k} \mathcal{H}^{n-1}(\partial E_i) \\
&\leq \sum_{j=1}^{p} \mathcal{H}^{n-1}(\partial E_{i_j}) + \sum_{j=p+1}^{k} \mathcal{H}^{n-1}(\partial E_{i_j}) \\
&\leq \sum_{j=1}^{p} \Big(\mathcal{H}^{n-1}(\partial(\operatorname{co}(E))) - \delta(\operatorname{co}(E), E_{i_j})\Big) + \sum_{j=p+1}^{k} \mathcal{H}^{n-1}(\partial(\operatorname{co}(E))) \\
&\leq k\mathcal{H}^{n-1}(\partial(\operatorname{co}(E))) - \sum_{j=1}^{p} \delta(\operatorname{co}(E), E_{i_j}),
\end{aligned}$$

so that

$$\left\lceil \frac{\mathcal{H}^{n-1}(\partial E) + \sum_{j=1}^{p} \delta(\operatorname{co}(E), E_{i_j})}{\mathcal{H}^{n-1}(\partial(\operatorname{co}(E)))} \right\rceil \leq k.$$

Now, since $E_{i_j} \subset H_{i_j}$, we observe that

$$\begin{aligned}
\delta(\operatorname{co}(E), E_{i_j}) &= \mathcal{H}^{n-1}(\partial(\operatorname{co}(E))) - \mathcal{H}^{n-1}(\partial E_{i_j}) \\
&= \Big(\mathcal{H}^{n-1}(\partial(\operatorname{co}(E))) - \mathcal{H}^{n-1}(\partial(\operatorname{co}(E) \cap H_{i_j}))\Big) \\
&\quad + \Big(\mathcal{H}^{n-1}(\partial(\operatorname{co}(E) \cap H_{i_j})) - \mathcal{H}^{n-1}(\partial E_{i_j})\Big) \\
&= \delta(\operatorname{co}(E) \cap H_{i_j}, E_{i_j}) + \delta(\operatorname{co}(E), \operatorname{co}(E) \cap H_{i_j}) \\
&\geq \delta(\operatorname{co}(E), \operatorname{co}(E) \cap H_{i_j})
\end{aligned}$$

for all $j = 1, \dots, p$. We can thus apply (1.2) to each couple of convex bodies $\mathrm{co}(E)$ and $\mathrm{co}(E) \cap H_{i_j}$, with $j = 1, \dots, p$, and get

$$\delta(\mathrm{co}(E), \mathrm{co}(E) \cap H_{i_j}) \geq \frac{\omega_{n-1} r_{i_j}^{n-2} h_{i_j}^2}{r_{i_j} + \sqrt{r_{i_j}^2 + h_{i_j}^2}}, \tag{3.1}$$

where

$$h_{i_j} = h(\mathrm{co}(E) \cap H_{i_j}, \mathrm{co}(E)), \qquad r_{i_j} = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(\mathrm{co}(E) \cap \partial H_{i_j})}{\omega_{n-1}}}.$$

By (2.6), we clearly have

$$\beta \rho_{\nu_{i_j}}(\mathrm{co}(E)) \leq r_{i_j} \leq \rho_{\nu_{i_j}}(\mathrm{co}(E)) \tag{3.2}$$

for all $j = 1, \dots, p$. Inserting (3.2) into (3.1), we immediately obtain that

$$\delta(\mathrm{co}(E) \cap H_{i_j}, \mathrm{co}(E)) \geq \frac{\omega_{n-1} \beta^{n-2} \rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-2} h_{i_j}^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + h_{i_j}^2}}$$

for all $j = 1, \dots, p$. Now, for any given $c > 0$, the function

$$s \mapsto \frac{s^2}{c + \sqrt{c + s^2}}$$

is strictly increasing for $s > 0$. Since $h_{i_j} \geq \alpha \, \mathrm{diam}(\mathrm{co}(E))$ for all $j = 1, \dots, p$ by (2.5), we can finally estimate

$$\delta(\mathrm{co}(E), E_{i_j}) \geq \frac{\omega_{n-1} \alpha^2 \beta^{n-2} \rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-2} \, \mathrm{diam}(\mathrm{co}(E))^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + \alpha^2 \, \mathrm{diam}(\mathrm{co}(E))^2}}$$

for all $j = 1, \dots, p$. In conclusion, we get

$$\begin{aligned} k &\geq \left\lceil \frac{\mathcal{H}^{n-1}(\partial E) + \sum_{j=1}^{p} \delta(E_{i_j}, \mathrm{co}(E))}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(E)))} \right\rceil \\ &\geq \left\lceil \frac{\mathcal{H}^{n-1}(\partial E) + \omega_{n-1} \alpha^2 \beta^{n-2} \sum_{j=1}^{p} \frac{\rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-2} \, \mathrm{diam}(\mathrm{co}(E))^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + \alpha^2 \, \mathrm{diam}(\mathrm{co}(E))^2}}}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(E)))} \right\rceil \end{aligned}$$

proving (2.7). The proof is thus complete. □

4. Examples

4.1. **An example in $\mathbb{R}^2$.** We begin with the following example in $\mathbb{R}^2$ showing that our Theorem 2.4 in the planar formulation (2.8), at least in some cases, provides a strictly better estimate than the one in (2.4) previously established in [6]. This example is based on the set $C \subset \mathbb{R}^2$ shown in Figure 2, which was already considered in [9, Example 2.1] and in [6, Example 3.1]. The set C depends on two parameters $l > h > 0$. In [6, Example 3.1], to make the construction work, it was necessary to assume that $h \in (0, \varepsilon)$ for some $\varepsilon \in (0, l)$ sufficiently small. In our situation, thanks to the refined inequality (2.8), our choice of the parameter h is less restrictive, i.e., we are going to choose $h \in (0, \bar{\varepsilon})$ for some $\bar{\varepsilon} \in (\varepsilon, l)$. As matter of fact, when $h \in (\varepsilon, \bar{\varepsilon})$, our inequality (2.8) gives the correct value $k_{\min}(C) = 3$, while inequality (2.4) gives the lower bound $k_{\min}(C) \geq 2$ only.

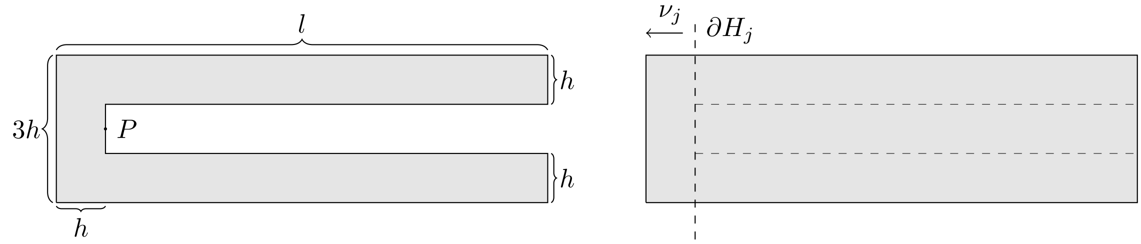

FIGURE 2. The set $C \subset \mathbb{R}^2$ (on the left) and its convex hull (on the right).

Example 4.1 (The set $C \subset \mathbb{R}^2$)**.** Let $l > h > 0$ and consider the set $C \subset \mathbb{R}^2$ in Figure 2. We can compute

$$\mathcal{H}^1(\partial C) = 4l + 4h, \quad \mathcal{H}^1(\partial(\mathrm{co}(C))) = 2l + 6h, \quad \mathrm{diam}(\mathrm{co}(C)) = \sqrt{l^2 + 9h^2}.$$

Since C is not convex, we must have that $k_{\min}(C) \geq 2$. After all, it is evident that $k_{\min}(C) = 3$. Our argument will give such right value for a larger class of parameters $l > h > 0$ than the one provided in [6, Example 3.1]. First of all, notice that we do not deduce any further information from the result in [9]. Indeed, inequality (2.2) only yields

$$k_{\min}(C) \geq \left\lceil \frac{\mathcal{H}^1(\partial C)}{\mathcal{H}^1(\partial(\mathrm{co}(C)))} \right\rceil = 2,$$

since an elementary computation shows that

$$\frac{\mathcal{H}^1(\partial C)}{\mathcal{H}^1(\partial(\mathrm{co}(C)))} = \frac{2l + 2h}{l + 3h} \in (1, 2)$$

whenever $l > h > 0$. Now we consider the point $P \in \partial C$ as shown in Figure 2. For every decomposition of C into convex bodies, there exists a convex body E_j containing P. Since E_j is convex and contained in C, we must have that $E_j \subset H_j$, where H_j is the half-space such that ∂H_j contains the face of C to which the point P belongs, see Figure 2. Consequently, we must have

$$h(\mathrm{co}(C) \cap H_j, \mathrm{co}(C)) = l - h, \quad \mathcal{H}^1(\mathrm{co}(C) \cap \partial H_j) = 3h, \quad \rho_{\nu_j}(\mathrm{co}(C)) = \frac{3h}{2},$$

where $\nu_j \in \mathbb{S}^1$ is the inner unit normal of the half-space H_j as in Figure 2. Now let $l > 0$ be fixed. In [6], it has been shown that, for any $\alpha \in (0, 1)$, $p = 1$ and $h \ll l$, one has

$$\left\lceil \frac{\mathcal{H}^1(\partial C) + \frac{4\alpha^2}{1+\sqrt{1+4\alpha^2}} \,\mathrm{diam}(\mathrm{co}(C))}{\mathcal{H}^1(\partial(\mathrm{co}(C)))} \right\rceil = 3.$$

We now apply inequality (2.4) and Theorem 2.4 with

$$p = 1, \quad \alpha = \frac{l - h}{\sqrt{l^2 + 9h^2}}, \quad \beta = 0.$$

We claim that we can choose $h \in (0,l)$ such that

$$\left\lceil \frac{\mathcal{H}^1(\partial C) + \frac{4\alpha^2}{1+\sqrt{1+4\alpha^2}}\,\mathrm{diam}(\mathrm{co}(C))}{\mathcal{H}^1(\partial(\mathrm{co}(C)))} \right\rceil = 2$$

and

$$\left\lceil \frac{\mathcal{H}^1(\partial C) + \frac{2\alpha^2\,\mathrm{diam}(\mathrm{co}(C))^2}{\rho_{\nu_j}(\mathrm{co}(C))+\sqrt{\rho_{\nu_j}(\mathrm{co}(C))^2+\alpha^2\,\mathrm{diam}(\mathrm{co}(C))^2}}}{\mathcal{H}^1(\partial(\mathrm{co}(C)))} \right\rceil = 3.$$

In order to have both the claimed inequalities, it is sufficient to find $h \in (0,l)$ such that

$$\frac{\mathcal{H}^1(\partial C) + \frac{4\alpha^2}{1+\sqrt{1+4\alpha^2}}\,\mathrm{diam}(\mathrm{co}(C))}{\mathcal{H}^1(\partial(\mathrm{co}(C)))} \le 2 < \frac{\mathcal{H}^1(\partial C) + \frac{2\alpha^2\,\mathrm{diam}(\mathrm{co}(C))^2}{\rho_{\nu_j}(\mathrm{co}(C))+\sqrt{\rho_{\nu_j}(\mathrm{co}(C))^2+\alpha^2\,\mathrm{diam}(\mathrm{co}(C))^2}}}{\mathcal{H}^1(\partial(\mathrm{co}(C)))},$$

that is,

$$\frac{2(l+h) + \frac{2\alpha^2}{1+\sqrt{1+4\alpha^2}}\sqrt{l^2+9h^2}}{l+3h} \le 2 < \frac{2(l+h) + \frac{2\alpha^2(l^2+9h^2)}{3h+\sqrt{9h^2+4\alpha^2(l^2+9h^2)}}}{l+3h}.$$

Up to some elementary algebraic computations, we need to find $h \in (0,l)$ such that

$$\frac{(l-h)^2}{3h+\sqrt{9h^2+4(l-h)^2}} > 2h \ge \frac{(l-h)^2}{\sqrt{l^2+9h^2}+\sqrt{l^2+9h^2+4(l-h)^2}}.$$

If we let $h = tl$ for $t \in (0,1)$, then we just need to solve

$$\begin{cases} 1-5t^2-2t-2t\sqrt{9t^2+4(1-t)^2} > 0 \\ 2t\sqrt{1+9t^2}+2t\sqrt{1+9t^2+4(1-t)^2}-1-t^2+2t \ge 0 \end{cases}$$

and we let the reader check that the above system of inequalities admits solutions.

4.2. **Two examples in $\mathbb{R}^3$.** We now give two examples in $\mathbb{R}^3$ showing that for $n = 3$ our Theorem 2.4 provides an improvement of the inequality (2.2) established in [9].

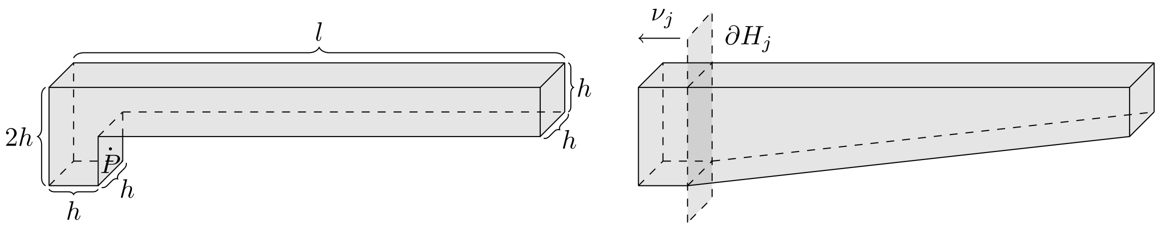

FIGURE 3. The set $L \subset \mathbb{R}^3$ (on the left) and its convex hull (on the right).

Example 4.2 (The body $L \subset \mathbb{R}^3$)**.** Let $l > h > 0$ and consider the set $L \subset \mathbb{R}^3$ in Figure 3. We can compute

$$\begin{aligned} \mathcal{H}^2(\partial L) &= 4hl + 6h^2, \\ \mathcal{H}^2(\partial(\mathrm{co}(L))) &= 4hl + 5h^2 + h\sqrt{(l-h)^2+h^2}, \\ \mathrm{diam}(\mathrm{co}(L)) &= \sqrt{l^2+5h^2}. \end{aligned}$$

Since L is not convex, we must have that $k_{\min}(L) \geq 2$, and a simple geometric argument allows to conclude that $k_{\min}(L) = 2$. From (2.2) we deduce that

$$k_{\min}(L) = 2 > 1 = \left\lceil \frac{\mathcal{H}^2(\partial L)}{\mathcal{H}^2(\partial(\mathrm{co}(L)))} \right\rceil,$$

since an elementary computation shows that

$$\frac{\mathcal{H}^2(\partial L)}{\mathcal{H}^2(\partial(\mathrm{co}(L)))} = \frac{4l + 6h}{4l + 5h + \sqrt{(l-h)^2 + h^2}} \in (0,1)$$

whenever $l > h > 0$. Now we consider the point $P \in \partial L$ as shown in Figure 3. For every decomposition of L into convex bodies, there exists a convex body E_j containing P. Since E_j is convex and contained in L, we must have that $E_j \subset H_j$, where H_j is the half-space such that ∂H_j contains the face of L to which the point P belongs, see Figure 3. Consequently, we must have

$$h(\mathrm{co}(L) \cap H_j, \mathrm{co}(L)) = l - h, \quad \mathcal{H}^2(\mathrm{co}(L) \cap \partial H_j) = 2h^2, \quad \rho_{\nu_j}(\mathrm{co}(L)) = \sqrt{\frac{2h^2}{\pi}},$$

where $\nu_j \in \mathbb{S}^2$ is the inner unit normal of the half-space H_j as in Figure 3. We now let $l > 0$ be fixed. We apply Theorem 2.4 with

$$p = 1, \quad \alpha = \frac{l-h}{\sqrt{l^2 + 5h^2}}, \quad \beta = 1.$$

Provided that we choose $h \in (0, l)$ sufficiently small, we conclude that

$$k_{\min}(L) \geq \left\lceil \frac{4hl + 6h^2 + \pi \left(\frac{l-h}{\sqrt{l^2+5h^2}}\right)^2 \frac{\sqrt{\frac{2h^2}{\pi}} \left(\sqrt{l^2 + 5h^2}\right)^2}{\sqrt{\frac{2h^2}{\pi}} + \sqrt{\frac{2h^2}{\pi} + \left(\frac{l-h}{\sqrt{l^2+5h^2}}\right)^2 \left(\sqrt{l^2+5h^2}\right)^2}}}{4hl + 5h^2 + h\sqrt{(l-h)^2 + h^2}} \right\rceil$$

$$= \left\lceil \frac{4l + 6h + \frac{\sqrt{2\pi}\,(l-h)^2}{\sqrt{\frac{2h^2}{\pi}} + \sqrt{\frac{2h^2}{\pi} + (l-h)^2}}}{4l + 5h + \sqrt{(l-h)^2 + h^2}} \right\rceil = 2,$$

since

$$\lim_{h \to 0^+} \frac{4l + 6h + \frac{\sqrt{2\pi}\,(l-h)^2}{\sqrt{\frac{2h^2}{\pi}} + \sqrt{\frac{2h^2}{\pi} + (l-h)^2}}}{4l + 5h + \sqrt{(l-h)^2 + h^2}} = \frac{4 + \sqrt{2\pi}}{5} \in (1, 2).$$

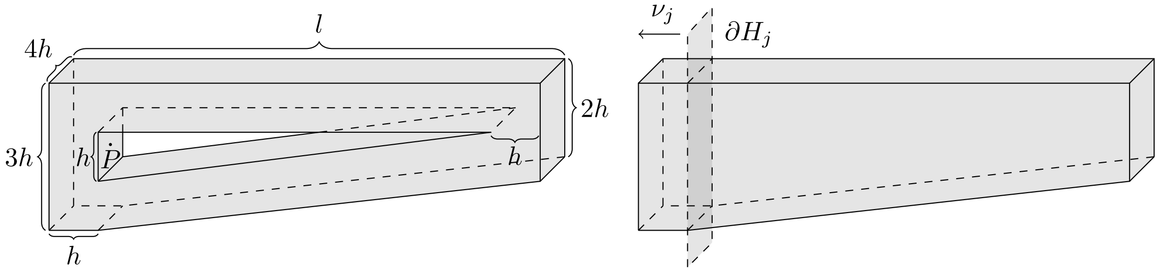

FIGURE 4. The set $D \subset \mathbb{R}^3$ (on the left) and its convex hull (on the right).

Example 4.3 (The set D in $\mathbb{R}^3$)**.** Let $l > 2h > 0$ and consider the set $D \subset \mathbb{R}^3$ in Figure 4. We can compute

$$\begin{aligned}\mathcal{H}^2(\partial D) &= 12lh + 4h\sqrt{(l-h)^2+h^2} + 4h\sqrt{(l-2h)^2+h^2} + 23h^2,\\ \mathcal{H}(\partial(\mathrm{co}(D))) &= 9lh + 4h\sqrt{(l-h)^2+h^2} + 25h^2,\\ \mathrm{diam}(\mathrm{co}(D)) &= \sqrt{l^2+25h^2}.\end{aligned}$$

Since D is not convex, we must have that $k_{\min}(D) \geq 2$, and a simple geometric argument allows to conclude that $k_{\min}(D) \geq 3$. From (2.2) we deduce that

$$k_{\min}(D) = 3 > 2 = \left\lceil \frac{\mathcal{H}^2(\partial D)}{\mathcal{H}^2(\partial(\mathrm{co}(D)))} \right\rceil,$$

since an elementary computation shows that

$$\frac{\mathcal{H}^2(\partial D)}{\mathcal{H}^2(\partial(\mathrm{co}(D)))} = \frac{12l + 4\sqrt{(l-h)^2+h^2} + 4\sqrt{(l-2h)^2+h^2} + 23h}{9l + 4\sqrt{(l-h)^2+h^2} + 25h} \in (1,2)$$

whenever $l > 2h > 0$. Now we consider the point $P \in \partial D$ as shown in Figure 4. For every decomposition of D into convex bodies, there exists a convex body E_j containing P. Since E_j is convex and contained in D, we must have that $E_j \subset H_j$, where H_j is the half-space such that ∂H_j contains the face of D to which the point P belongs, see Figure 4. Consequently, we must have

$$h(\mathrm{co}(D) \cap H_j, \mathrm{co}(D)) = l - h, \quad \mathcal{H}^2(\mathrm{co}(D) \cap \partial H_j) = 12h^2, \quad \rho_{\nu_j}(\mathrm{co}(D)) = \sqrt{\frac{12h^2}{\pi}},$$

where $\nu_j \in \mathbb{S}^2$ is the inner unit normal of the half-space H_j as in Figure 4. We now let $l > 0$ be fixed. We apply Theorem 2.4 with

$$p = 1, \quad \alpha = \frac{l-h}{\sqrt{l^2+25h^2}}, \quad \beta = 1.$$

Provided that we choose $h \in \left(0, \frac{l}{2}\right)$ sufficiently small, we conclude that

$$k_{\min}(D) \geq \left\lceil \frac{\mathcal{H}^2(\partial D) + \pi\left(\frac{l-h}{\sqrt{l^2+25h^2}}\right)^2 \frac{\sqrt{\frac{12h^2}{\pi}}\left(\sqrt{l^2+25h^2}\right)^2}{\sqrt{\frac{12h^2}{\pi}+\sqrt{\frac{12h^2}{\pi}+\left(\frac{l-h}{\sqrt{l^2+25h^2}}\right)^2\left(\sqrt{l^2+25h^2}\right)^2}}}}{\mathcal{H}^2(\partial(\mathrm{co}(D)))} \right\rceil$$

$$= \left\lceil \frac{12l + 4\sqrt{(l-h)^2+h^2} + 4\sqrt{(l-2h)^2+h^2} + 23h + \frac{\sqrt{12\pi}\,(l-h)^2}{\sqrt{\frac{12h^2}{\pi}+\sqrt{\frac{12h^2}{\pi}+(l-h)^2}}}}{9l + 4\sqrt{(l-h)^2+h^2} + 25h} \right\rceil = 3,$$

since

$$\lim_{h\to 0^+} \frac{12l + 4\sqrt{(l-h)^2+h^2} + 4\sqrt{(l-2h)^2+h^2} + 23h + \frac{\sqrt{12\pi}\,(l-h)^2}{\sqrt{\frac{12h^2}{\pi}+\sqrt{\frac{12h^2}{\pi}+(l-h)^2}}}}{9l + 4\sqrt{(l-h)^2+h^2} + 25h}$$
$$= \frac{20+\sqrt{12\pi}}{13} \in (2,3).$$

4.3. **An example in $\mathbb{R}^n$.** We conclude this section with Example 4.5 below showing that for all $n \geq 3$ our Theorem 2.4 provides an improvement of the inequality (2.2) established in [9]. In Example 4.5 we will need to apply the following result, whose elementary proof is detailed below for the reader's convenience.

Lemma 4.4. *Let $\ell \in (0,+\infty)$ and let $Q \subset \mathbb{R}^2$ be a set with*

$$\mathcal{H}^1(\partial Q) < +\infty \quad \text{and} \quad \mathcal{H}^2(Q) < +\infty.$$

If $E_n = Q \times [0,\ell]^{n-2} \subset \mathbb{R}^n$, then

$$\mathcal{H}^{n-1}(\partial E_n) = \ell^{n-2}\,\mathcal{H}^1(\partial Q) + 2(n-2)\,\ell^{n-3}\,\mathcal{H}^2(Q) \tag{4.1}$$

for all $n \geq 2$.

Proof. By definition, the set $E_n \subset \mathbb{R}^n$ satisfies

$$\mathcal{H}^n(E_n) = \ell^{n-2}\,\mathcal{H}^2(Q). \tag{4.2}$$

Moreover, since we can recursively write $E_n = E_{n-1} \times [0,\ell]$ and thus

$$\partial E_n = ((\partial E_{n-1}) \times [0,\ell]) \cup (E_{n-1} \times \{0,\ell\}),$$

by the coarea formula we can compute

$$\mathcal{H}^{n-1}(\partial E_n) = 2\mathcal{H}^{n-1}(E_{n-1}) + \ell\,\mathcal{H}^{n-2}(\partial E_{n-1})$$

for all $n \geq 2$. The validity of (4.1) can thus be checked by induction, thanks to (4.2). $\square$

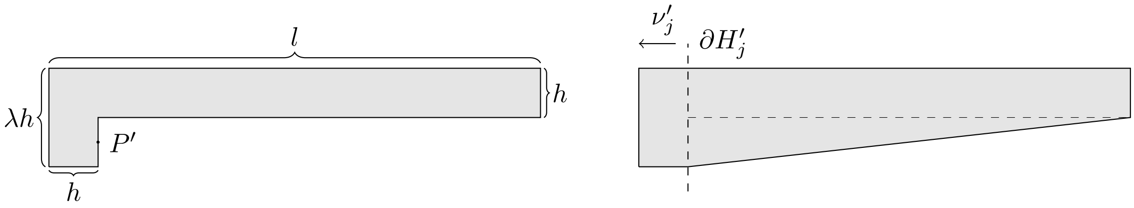

FIGURE 5. The body $L_2 \subset \mathbb{R}^2$ (on the left) and its convex hull (on the right).

Example 4.5 (The set $L_n \subset \mathbb{R}^n$ for $n \geq 3$)**.** Let $l > h > 0$ and $\lambda > 1$ and consider the set $L_n = L_2 \times [0,h]^{n-2} \subset \mathbb{R}^n$ for $n \geq 3$, where $L_2 \subset \mathbb{R}^n$ is the set in Figure 5. Note that

$$\mathcal{H}^1(\partial L_2) = 2l + 2\lambda h, \quad \mathcal{H}^2(L_2) = h\Big(l + (\lambda-1)h\Big)$$

and, similarly,

$$\begin{aligned}\mathcal{H}^1(\partial(\mathrm{co}(L_2))) &= l + \sqrt{(l-h)^2 + (\lambda-1)^2h^2} + (\lambda+2)h,\\ \mathcal{H}^2(\mathrm{co}(L_2)) &= \frac{h}{2}\Big((\lambda+1)l + (\lambda-1)h\Big).\end{aligned}$$

Since $\mathrm{co}(L_n) = \mathrm{co}(L_2) \times [0,h]^{n-2}$, we can apply Lemma 4.4 to compute

$$\begin{aligned}\mathcal{H}^{n-1}(\partial L_n) &= 2h^{n-2}\Big((n-1)l + ((n-1)\lambda - n + 2))h\Big),\\ \mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n))) &= h^{n-2}\Big(((n-2)\lambda + n - 1)l + \sqrt{(l-h)^2 + (\lambda-1)^2h^2}\\ &\qquad + ((n-1)\lambda - n + 4)h\Big),\\ \mathrm{diam}(\mathrm{co}(L_n)) &= \sqrt{l^2 + (\lambda^2 + n - 2)h^2}\end{aligned}$$

for all $n \geq 3$. Note that L_n is not convex, so we must have that $k_{\min}(L_n) \geq 2$ for all $n \geq 3$. In fact, a simple geometric decomposition proves that $k_{\min}(L_n) = 2$ for all $n \geq 3$. Now we consider the point $P = (P', 0) \in L_n$, where $P' \in \partial L_2$ is shown in Figure 5. For every decomposition of L_n into convex bodies, there exists a convex body E_j containing P. Since E_j is convex and contained in L_n, we must have that its projection $E_j' = \mathsf{P}_{\mathbb{R}^2}(E_j)$ is a convex body contained in in $L_2 \cap H_j'$, where $\mathsf{P}_{\mathbb{R}^2} \colon \mathbb{R}^n \to \mathbb{R}^2$ is the canonical projection onto the first two coordinates and H_j' is the half-plane such that $\partial H_j'$ contains the face of L_2 to which the point P belongs, see Figure 5. Therefore, we must have that $E_j \subset H_j$, where H_j is the half-space $H_j = \mathsf{P}_{\mathbb{R}^2}^{-1}(H_j') \subset \mathbb{R}^n$. Consequently, we must have

$$h(\mathrm{co}(L_n) \cap H_j, \mathrm{co}(L_n)) = l - h, \quad \mathcal{H}^{n-1}(\mathrm{co}(L_n) \cap \partial H_j) = \lambda h^{n-1}, \quad \rho_{\nu_j}(\mathrm{co}(L_n)) = \sqrt[n-1]{\frac{\lambda h^{n-1}}{\omega_{n-1}}},$$

where $\nu_j \in \mathbb{S}^{n-1}$ is the inner unit normal of the half-space H_j (precisely, $\nu_j = (\nu_j', 0)$, where ν_j' is the inner unit normal of H_j', see Figure 5). Now we let $l > 0$ be fixed. We apply Theorem 2.4 with

$$p = 1, \quad \alpha = \frac{l-h}{\sqrt{l^2 + (\lambda^2 + n - 2)h^2}}, \quad \beta = 1.$$

We are going to choose $\lambda > 1$ as a dimensional constant and $h \in (0, l)$ sufficiently small. Indeed, for any given $\lambda > 1$, we have that

$$\lim_{h\to 0^+} \frac{\mathcal{H}^{n-1}(\partial L_n)}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n)))} = \frac{2n-2}{(n-2)\lambda + n}$$

and, similarly,

$$\lim_{h\to 0^+} \frac{\mathcal{H}^{n-1}(\partial L_n) + \omega_{n-1}\alpha^2\beta^{n-2} \frac{\rho_{\nu_{i_j}}(\mathrm{co}(L_n))^{n-2}\,\mathrm{diam}(\mathrm{co}(L_n))^2}{\rho_{\nu_j}(\mathrm{co}(L_n)) + \sqrt{\rho_{\nu_j}(\mathrm{co}(E))^2 + \alpha^2\,\mathrm{diam}(\mathrm{co}(L_n))^2}}}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n)))} = \frac{2n-2+c_n\lambda^{\frac{n-2}{n-1}}}{(n-2)\lambda + n},$$

where $c_n = \omega_{n-1}^{\frac{1}{n-1}} > 0$ is a dimensional constant. Since $\lambda > 1$, we have that

$$\frac{2n-2}{(n-2)\lambda+n} < 1 \quad \text{for all } n \geq 3.$$

On the other hand, we obviously have

$$\frac{2n-2+c_n\lambda^{\frac{n-2}{n-1}}}{(n-2)\lambda + n} > 1 \iff \lambda^{\frac{n-2}{n-1}} > \frac{n-2}{c_n}(\lambda - 1)$$

and it is possible to verify that the last inequality admits solutions in the interval $(1, +\infty)$. Consequently, for each $n \geq 3$ we can find $\lambda_n \in (1, +\infty)$ such that

$$\frac{2n-2+c_n\lambda_n^{\frac{n-2}{n-1}}}{(n-2)\lambda_n + n} > 1.$$

Therefore, provided that we choose $\lambda = \lambda_n$ as above and $h \in (0, l)$ sufficiently small, we conclude that the set $L_n \subset \mathbb{R}^n$ corresponding to these choices of parameters satisfies

$$\left\lceil \frac{\mathcal{H}^{n-1}(\partial L_n)}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n)))} \right\rceil = 1$$

and

$$\left\lceil \frac{\mathcal{H}^{n-1}(\partial L_n) + \omega_{n-1}\alpha^2\beta^{n-2} \frac{\rho_{\nu_{i_j}}(\mathrm{co}(L_n))^{n-2}\,\mathrm{diam}(\mathrm{co}(L_n))^2}{\rho_{\nu_j}(\mathrm{co}(L_n)) + \sqrt{\rho_{\nu_j}(\mathrm{co}(E))^2 + \alpha^2\,\mathrm{diam}(\mathrm{co}(L_n))^2}}}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n)))} \right\rceil = 2.$$

References

[1] Archimedes, *The works of Archimedes. Vol. I*, Cambridge University Press, Cambridge, 2004. The two books on the sphere and the cylinder; Translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams by Reviel Netz.

[2] T. Bonnesen and W. Fenchel, *Theory of convex bodies*, BCS Associates, Moscow, ID, 1987. Translated from the German and edited by L. Boron, C. Christenson and B. Smith.

[3] G. Buttazzo, V. Ferone, and B. Kawohl, *Minimum problems over sets of concave functions and related questions*, Math. Nachr. **173** (1995), 71–89.

[4] M. Carozza, F. Giannetti, F. Leonetti, and A. Passarelli di Napoli, *A sharp quantitative estimate for the perimeters of convex sets in the plane*, J. Convex Anal. **22** (2015), no. 3, 853–858.

[5] ———, *A sharp quantitative estimate for the surface areas of convex sets in* $\mathbb{R}^3$, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **27** (2016), no. 3, 327–333.

[6] ———, *Convex components*, Commun. Contemp. Math. **21** (2019), no. 6, 1850036, 10.

[7] F. Giannetti and G. Stefani, *On the convex components of a set in* $\mathbb{R}^n$ (2021), preprint, available at https://arxiv.org/abs/2106.08725.

[8] P. M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 336, Springer, Berlin, 2007.

[9] M. La Civita and F. Leonetti, *Convex components of a set and the measure of its boundary*, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia **56** (2008/09), 71–78.

[10] F. Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[11] G. Stefani, *On the monotonicity of perimeter of convex bodies*, J. Convex Anal. **25** (2018), no. 1, 93–102.

(G. Stefani) Department Mathematik und Informatik, Universität Basel, Spiegelgasse 1, CH-4051 Basel, Switzerland

Email address: giorgio.stefani@unibas.ch