

An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces

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Euler equations, velocity form

The **Euler equations** for an incompressible inviscid 2-dimensional fluid are given by

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla p = 0 & \text{in } (0, +\infty) \times \Omega, \\ \operatorname{div} v = 0 & \text{in } [0, +\infty) \times \Omega, \\ v \cdot \nu_\Omega = 0 & \text{on } [0, +\infty) \times \partial\Omega, \\ v|_{t=0} = v_0 & \text{on } \Omega. \end{cases}$$

Objects:

- Ω is a sufficiently smooth (possibly unbounded) open set or the flat torus \mathbb{T}^2 ;
- $v: [0, +\infty) \times \Omega \rightarrow \mathbb{R}^2$ is the **velocity** of the fluid;
- $p: [0, +\infty) \times \Omega \rightarrow \mathbb{R}$ is the (scalar) **pressure**;
- $\nu_\Omega: \partial\Omega \rightarrow \mathbb{R}^2$ is the inner unit **normal** to $\partial\Omega$.

Conditions:

- $\operatorname{div} v = 0$ is the **incompressibility** condition;
- $v \cdot \nu_\Omega = 0$ at the boundary is the **no-flow** (or **slip**) condition.

Note: either $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{T}^2 \Rightarrow$ no boundary condition is imposed.

Euler equations, vorticity form

The **vorticity** $\omega: [0, +\infty) \times \Omega \rightarrow \mathbb{R}$ of the fluid is

$$\omega = \operatorname{curl} v$$

and satisfies

$$\begin{cases} \partial_t \omega + \operatorname{div}(v\omega) = 0 & \text{in } (0, +\infty) \times \Omega, \\ v = K\omega & \text{in } [0, +\infty) \times \Omega, \\ \omega|_{t=0} = \omega_0 & \text{on } \Omega. \end{cases}$$

Biot-Savart law: The relation $\omega = K v$ is the **Biot-Savart law**, i.e.

$$v(t, x) = K\omega(t, x) = \int_{\Omega} k(x, y) \omega(t, y) dy,$$

where $k: \Omega \times \Omega \rightarrow \mathbb{R}^2$ is a convolution kernel.

Example: If $\Omega = \mathbb{R}^2$, then $k(x, y) = k_2(x - y)$ with

$$k_2(x) = \frac{1}{2\pi} \frac{1}{|x|^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} \quad \text{for all } x \in \mathbb{R}^2, x \neq 0.$$

Literature: a quick review

Theory of strong solutions is classical (since Lichtenstein 1930).

Existence of weak solutions:

- Yudovich (1963) for $L^1 \cap L^\infty$ vorticity
- DiPerna-Majda (1987), Delort (1991), Majda (1993), Vecchi-Wu (1993), Evans-Müller (1994) for L^1 vorticity
- Serfati (1995), Vishik (1999), Taniuchi (2004) for **non-decaying** vorticity

Uniqueness of weak solutions:

- Yudovich (1963) for $L^1 \cap L^\infty$ vorticity
- Yudovich (1995) for **unbounded** vorticity with L^p -norm mildly growing
- Vishik (1999) for ∞ -Besov vorticity

Philosophy: while **existence** follows the usual pattern

smoothing data \rightarrow existence of smooth solutions \rightarrow compactness,

uniqueness is hard, due to non-linearity of Euler equations.

Warning: **uniqueness** is NOT expected for vorticity in L^p with $p < +\infty$!

- Vishik (2018)
- Bressan-Murray (2020), Bressan-Shen (2021)
- Bruè-Colombo (2021)

Yudovich's well-posedness for $L^1 \cap L^\infty$

Recall the Euler 2D equations in vorticity form:

$$\begin{cases} \partial_t \omega + \operatorname{div}(v\omega) = 0 & \text{in } (0, +\infty) \times \Omega, \\ v = K\omega & \text{in } [0, +\infty) \times \Omega, \\ \omega|_{t=0} = \omega_0 & \text{on } \Omega. \end{cases} \quad (\text{E})$$

Theorem (Yudovich 1963)

There is a unique weak solution (ω, v) of (E) such that

$$\omega \in L^\infty([0, +\infty); L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \quad v \in L^\infty([0, +\infty); C_b(\mathbb{R}^2; \mathbb{R}^2))$$

starting from $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $v_0 = K\omega_0$. Actually, the velocity satisfies

$$|v(t, x) - v(t, y)| \lesssim |x - y| \cdot |\log|x - y|| \quad x, y \in \mathbb{R}^2, t \geq 0.$$

Proof:

- **existence** relies on compactness of smooth solutions;
- **uniqueness** follows from a clever energy method.

Yudovich's energy method

Assume (ω^1, v^1) and (ω^2, v^2) are two weak solutions with same initial datum.

Consider the **relative energy**

$$E(t) = \int_{\mathbb{R}^2} |v^1(t, x) - v^2(t, x)|^2 dx \quad \text{for } t \geq 0.$$

Since $v = K\omega$, then $\nabla v = (\nabla K)\omega$, with ∇K a **Calderón-Zygmund operator**. Hence

$$\|\nabla v\|_{L^p} \lesssim p \|\omega\|_{L^p} \leq Cp \quad \text{for all } p \gg 1$$

where $C > 0$ depends on $\|\omega\|_{L^1}$ and $\|\omega\|_{L^\infty}$ only.

Exploit the Euler equations in **velocity form** $\partial_t v + (v \cdot \nabla)v + \nabla p = 0$ to get

$$\frac{d}{dt} E(t) \leq Cp E(t)^{1-1/p} \quad \text{for } t \in [0, T],$$

where $T > 0$ has to be chosen.

By comparison with the maximal solution of the ODE, we get

$$E(t) \leq (Ct)^p \leq (CT)^p \quad \text{for } t \in [0, T],$$

so that $E(t) = 0$ for all $t \in [0, T]$ **letting $p \rightarrow +\infty$** , provided that $CT < 1$.

Yudovich's energy method revised

If $\|\omega\|_{L^p} \lesssim \Theta(p)$ for $p \gg 1$ for some **growth function** $\Theta: [1, +\infty) \rightarrow (0, +\infty)$, then

$$\|\nabla v\|_{L^p} \lesssim p \|\omega\|_{L^p} \lesssim p \Theta(p) \quad \text{for all } p \gg 1.$$

Re-doing the same computations, one finds

$$\frac{d}{dt} E(t) \lesssim E(t) \psi_{\Theta} \left(\frac{1}{E(t)} \right) \quad \text{for } t \in [0, T],$$

where

$$\psi_{\Theta}(r) = \begin{cases} \inf \left\{ \frac{1}{\varepsilon} \Theta(1/\varepsilon) : \varepsilon \in (0, 1/3) \right\} & \text{for } r \in [0, 1], \\ \inf \left\{ \frac{1}{\varepsilon} \Theta(1/\varepsilon) r^{\varepsilon} : \varepsilon \in (0, 1/3) \right\} & \text{for } r \in [1, +\infty). \end{cases}$$

To show $E(t) = 0$ for $t \in [0, T]$, we just need $z \mapsto z \psi_{\Theta}(1/z)$ to satisfy

$$\int_{0+} \frac{dr}{r \psi_{\Theta}(1/r)} = +\infty,$$

the well-known **Osgood condition**.

Yudovich's well-posedness for Y^Θ

We let

$$Y^\Theta(\Omega) = \left\{ f \in \bigcap_{p \in [1, +\infty)} L^p(\Omega) : \|f\|_{Y^\Theta(\Omega)} = \sup_{p \in [1, +\infty)} \frac{\|f\|_{L^p(\Omega)}}{\Theta(p)} < +\infty \right\}$$

be the **Yudovich space** on Ω associated to Θ .

Theorem (Yudovich 1995)

Assume Θ is such that

$$\int_{0+} \frac{dr}{r \psi_\Theta(1/r)} = +\infty.$$

There is a unique weak solution (ω, v) of (E) such that

$$\omega \in L^\infty([0, +\infty); Y^\Theta(\mathbb{R}^2)) \quad v \in L^\infty([0, +\infty); C_b(\mathbb{R}^2; \mathbb{R}^2))$$

starting from $\omega_0 \in Y^\Theta(\mathbb{R}^2)$, $v_0 = K\omega_0$. Actually, the velocity satisfies

$$|v(t, x) - v(t, y)| \lesssim |x - y| \cdot \psi_\Theta(1/|x - y|^3) \quad x, y \in \mathbb{R}^2, t \geq 0.$$

Examples of Θ

Bad news:

- energy method needs sharp tools (Sobolev spaces, CZ theory)
- behavior of ψ_Θ and its dependence on Θ are quite **implicit!**

Example: letting $\log_m p = \underbrace{\log \log \dots \log p}_{m \text{ times}}$, Yudovich proved that

$$\Theta_m(p) \approx \log p \log_2 p \cdots \log_m p \Rightarrow \psi_{\Theta_m}(r) \approx \log r \log_2 r \cdots \log_{m+1} r.$$

The Osgood condition is

- **true** for $\Theta(p) \approx \log p$ (actually, for any Θ_m with $m \geq 1$)
- **false** for $\Theta(p) \approx p$.

This means that Yudovich's result

- **applies** for vorticities with singularities of order $|\log |\log |x||$
- **does not apply** for vorticities with singularities of order $|\log |x||$ (e.g., BMO)

Question: is there a more **explicit** relation between Θ and the modulus of continuity?

Properties of the kernel: less is more

Recall the **Biot-Savart law** is given by (dropping time dependence)

$$v(x) = K\omega(x) = \int_{\Omega} k(x, y) \omega(y) dy.$$

The convolution kernel $k: \Omega \times \Omega \rightarrow \mathbb{R}^2$ satisfies

- **decay**: $|k(x, y)| \leq \frac{C_1}{|x - y|}$ for all $x, y \in \Omega, x \neq y$;
- **oscillation**: $|k(x, z) - k(y, z)| \leq C_2 \frac{|x - y|}{|x - z| |y - z|}$ for all $x, y, z \in \Omega, z \neq x, y$;

for some constants $C_1, C_2 > 0$.

From the relation $v = K\omega$, we also get

- **incompressibility**: $\operatorname{div}(K\omega) = 0$;
- **no-flow**: $(K\omega) \cdot \nu_{\Omega} = 0$ at the boundary.

IDEA: try to rely on the above 'metric' properties of k only!

A posteriori: we can even relax the incompressibility property to

- **controlled compression**: $\|\operatorname{div}(K\omega)\|_{L^{\infty}(\Omega)} \leq C_3 \|\omega\|_{L^1(\Omega)}$

for some constant $C_3 > 0$.

Exploit decay and oscillation

Fix $x, y \in \Omega$ with $d = |x - y| < 1$. We can split

$$\begin{aligned} |K\omega(x) - K\omega(y)| &\leq \int_{\Omega} |k(x, z) - k(y, z)| |\omega(z)| dz \\ &= \left(\int_{\Omega \setminus B_2(x)} + \int_{\Omega \cap (B_2(x) \setminus B_{2d}(x))} + \int_{\Omega \cap B_{2d}(x)} \right) |k(x, z) - k(y, z)| |\omega(z)| dz. \end{aligned}$$

We can estimate

$$\int_{\Omega \setminus B_2(x)} \dots \stackrel{\text{oscillation}}{\lesssim} |x - y| \int_{\Omega \setminus B_2(x)} \frac{|\omega(z)|}{|x - z| |y - z|} dz \lesssim |x - y| \|\omega\|_{L^1(\Omega)}$$

$$\int_{\Omega \cap (B_2(x) \setminus B_{2d}(x))} \dots \stackrel{\text{oscillation}}{\lesssim} \int_{\Omega \cap (B_2(x) \setminus B_{2d}(x))} \frac{|\omega(z)|}{|x - z|^2} dz$$

$$\int_{\Omega \cap B_{2d}(x)} \dots \stackrel{\text{decay}}{\lesssim} \int_{\Omega \cap B_{2d}(x)} \frac{|\omega(z)|}{|x - z|} dz + \int_{\Omega \cap B_{3d}(y)} \frac{|\omega(z)|}{|y - z|} dz$$

Two functions

We need to control

$$\alpha(d) = \sup_{x \in \Omega} \int_{\Omega \cap (B_2(x) \setminus B_{2d}(x))} \frac{|\omega(z)|}{|x-z|^2} dz \quad \text{and} \quad \beta(d) = \sup_{x \in \Omega} \int_{\Omega \cap B_{3d}(x)} \frac{|\omega(z)|}{|x-z|} dz$$

defined for $d \in (0, 1]$. By Hölder's inequality, we have

$$\begin{aligned} \alpha(d) &\lesssim \left(\sup_{x \in \Omega} \|\omega\|_{L^p(\Omega \cap B_2(x))} \right) \left(\int_{2d}^2 r^{1-2p'} dr \right)^{1/p'} \\ &\lesssim C \left(\frac{2^{2-2p'}}{2p'-2} \right)^{1/p'} \left(d^{2-2p'} - 1 \right)^{1/p'} \lesssim C p d^{-2/p} \end{aligned}$$

and, similarly,

$$\begin{aligned} \beta(d) &\lesssim \left(\sup_{x \in \Omega} \|\omega\|_{L^p(\Omega \cap B_3(x))} \right) \left(\int_0^{3d} r^{1-p'} dr \right)^{1/p'} \\ &\lesssim C \left(\frac{3^{2-p'}}{2-p'} \right)^{1/p'} d^{(2-p')/p'} \lesssim C \frac{p}{p-2} d^{1-2/p}, \end{aligned}$$

where $C = \sup_{x \in \Omega} \|\omega\|_{L^p(\Omega \cap B_1(x))}$.

Regularity of the velocity 1/3

We let

$$L_{\text{ul}}^p(\Omega) = \left\{ f \in L_{\text{loc}}^p(\Omega) : \|f\|_{L_{\text{ul}}^p(\Omega)} = \sup_{x \in \Omega} \|f\|_{L^p(\Omega \cap B_1(x))} < +\infty \right\}$$

be the **uniformly-localized L^p space** on Ω . Note that radius = 1 is not restrictive.

Theorem (Hölder continuity)

Let $p \in (2, +\infty)$. If $\omega \in L^1(\Omega) \cap L_{\text{ul}}^p(\Omega)$, then $K\omega \in C_b^{0,1-2/p}(\Omega; \mathbb{R}^2)$ with

$$\|K\omega\|_{L^\infty(\Omega; \mathbb{R}^2)} \lesssim \max\left\{1, \frac{1}{p-2}\right\} (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L_{\text{ul}}^p(\Omega)})$$

$$|K\omega(x) - K\omega(y)| \lesssim \max\left\{1, \frac{1}{p-2}\right\} (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L_{\text{ul}}^p(\Omega)}) p |x-y|^{1-2/p} \quad \forall x, y \in \Omega.$$

Remark: the result is **not** a surprise, since (for the Biot-Savart kernel)

CZ theory + Morrey's inequality \Rightarrow Hölder continuity.

However, our proof is surprising elementary!

Regularity of the velocity 2/3

We let

$$Y_{ul}^\Theta(\Omega) = \left\{ f \in \bigcap_{p \in [1, +\infty)} L_{ul}^p(\Omega) : \|f\|_{Y_{ul}^\Theta(\Omega)} = \sup_{p \in [1, +\infty)} \frac{\|f\|_{L_{ul}^p(\Omega)}}{\Theta(p)} < +\infty \right\}$$

be the **uniformly-localized Yudovich space** on Ω associated to Θ .

If $\omega \in L^1(\Omega) \cap Y_{ul}^\Theta(\Omega)$, then for all $p \geq 3$ we have

$$\begin{aligned} |K\omega(x) - K\omega(y)| &\lesssim \max\left\{1, \frac{1}{p-2}\right\} (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L_{ul}^p(\Omega)}) p |x - y|^{1-2/p} \\ &\lesssim (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y_{ul}^\Theta(\Omega)}) \Theta(p) p |x - y|^{1-2/p}. \end{aligned}$$

If $d = |x - y| \ll 1$, then we can take $p = |\log d| \gg 1$ and observe that

$$\Theta(p) p |x - y|^{1-2/p} = \Theta(|\log d|) |\log d| d^{1 - \frac{2}{|\log d|}} \approx d |\log d| \Theta(|\log d|)$$

since $d^{-\frac{2}{|\log d|}} = \exp\left(\frac{2}{\log d} \cdot \log d\right) = e^2$.

Regularity of the velocity 3/3

We let the function $\varphi_\Theta: [0, +\infty) \rightarrow [0, +\infty)$ be such that $\varphi_\Theta(0) = 0$ and

$$\varphi_\Theta(r) = \begin{cases} r(1 - \log r) \Theta(1 - \log r) & \text{for } r \in (0, e^{-2}] \\ e^{-2} 3 \Theta(3) & \text{for } r > e^{-2}. \end{cases}$$

We say that φ_Θ is the **modulus of continuity associated to Θ** and define

$$C_b^{0, \varphi_\Theta}(\Omega; \mathbb{R}^2) = \left\{ v \in L^\infty(\Omega; \mathbb{R}^2) : \sup_{x \neq y} \frac{|v(x) - v(y)|}{\varphi_\Theta(|x - y|)} < +\infty \right\}.$$

Corollary (φ_Θ -continuity)

If $\omega \in L^1(\Omega) \cap Y_{ul}^\Theta(\Omega)$, then $K\omega \in C_b^{0, \varphi_\Theta}(\Omega; \mathbb{R}^2)$ with

$$\|K\omega\|_{L^\infty(\Omega; \mathbb{R}^2)} \lesssim \|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y_{ul}^\Theta(\Omega)}$$

$$|K\omega(x) - K\omega(y)| \lesssim (\|\omega\|_{L^1(\Omega)} + \|\omega\|_{Y_{ul}^\Theta(\Omega)}) \varphi_\Theta(|x - y|) \quad \forall x, y \in \Omega.$$

Remark: we recover Yudovich's continuity modulus (e.g., Θ_m), with **NO** sharp tools!

Existence

By definition of φ_Θ , note that

$$\int_{0^+} \frac{dr}{\varphi_\Theta(r)} = \int^{+\infty} \frac{dp}{p\Theta(p)}.$$

Theorem (Existence)

Let $p \in (2, +\infty)$. For any $\omega_0 \in L^1(\Omega) \cap L_{ul}^p(\Omega)$, there is a weak solution (ω, v) of (E) such that

$$\omega \in L_{loc}^\infty([0, +\infty); L^1(\Omega) \cap L_{ul}^p(\Omega)) \quad v \in L_{loc}^\infty([0, +\infty); C_b^{0,1-2/p}(\Omega; \mathbb{R}^2)).$$

Moreover, if $\omega_0 \in L^1(\Omega) \cap Y_{ul}^\Theta(\Omega)$, then (ω, v) is such that

$$\omega \in L_{loc}^\infty([0, +\infty); L^1(\Omega) \cap Y_{ul}^\Theta(\Omega)) \quad v \in L_{loc}^\infty([0, +\infty); C_b^{0,\varphi_\Theta}(\Omega; \mathbb{R}^2))$$

and, provided that φ_Θ is Osgood, (ω, v) is Lagrangian.

ODE theory: φ_Θ Osgood \Rightarrow there is a unique flow X such that $\frac{d}{dt} X(t, \cdot) = v(t, X)$.

Lagrangian: the solution is such that $\omega(t, \cdot) = X(t, \cdot)_\# \omega_0$ (push-forward).

Remark: our existence result

- gives modulus of continuity even for Θ not BMO-like (e.g., $\Theta(p) \approx p^\alpha$);
- does not rely on the specific structure of the Biot-Savart kernel.

Proof of existence I/2

Warning: we **cannot** rely on the existence of smooth solutions!

Indeed, the kernel is general, so there are no equations in velocity form.

We have to follow a different strategy:

- 1) construct a solution in $L^1 \cap L^\infty$ via **time-stepping** argument;
- 2) construct a solution in $L^1 \cap L^p_{ul}$ by **truncating** the initial data;
- 3) show that the construction **preserves** the $L^1 \cap Y_{ul}^\Theta$ -regularity.

To gain existence, we need a **compactness criterion** à la Aubin-Lions:

- the proof exploits the Dunford-Pettis, Lusin and Arzelà-Ascoli Theorems;
- we assume **weak** compactness, while usually one takes **strong** compactness.

Proof of existence 2/2

Theorem (Baby Aubin-Lions)

Let $T > 0$ and let $(f^n)_{n \in \mathbb{N}} \subset L^\infty([0, T]; L^1(\Omega))$ be a **bounded** sequence which is **equi-integrable in space uniformly in time**:

- $\sup_{n \in \mathbb{N}} \|f^n\|_{L^\infty([0, T]; L^1(\Omega))} < +\infty$
- $\forall \varepsilon > 0 \exists \delta > 0 : A \subset \Omega, |A| < \delta \Rightarrow \sup_{n \in \mathbb{N}} \|f^n\|_{L^\infty([0, T]; L^1(A))} < \varepsilon$
- $\forall \varepsilon > 0 \exists \Omega_\varepsilon \subset \Omega$ with $|\Omega_\varepsilon| < +\infty : \sup_{n \in \mathbb{N}} \|f^n\|_{L^\infty([0, T]; L^1(\Omega \setminus \Omega_\varepsilon))} < \varepsilon.$

Assume that, for each $\varphi \in C_c^\infty(\Omega)$, the functions $F_n[\varphi]: [0, T] \rightarrow \mathbb{R}$, given by

$$F_n[\varphi](t) = \int_{\Omega} f^n(t, \cdot) \varphi \, dx, \quad t \in [0, T],$$

are **uniformly equi-continuous** on $[0, T]$.

Then there exist a subsequence $(f^{n_k})_{k \in \mathbb{N}}$ and $f \in L^\infty([0, T]; L^1(\Omega))$ such that

$$\lim_{k \rightarrow +\infty} \int_{\Omega} f^{n_k}(t, \cdot) \varphi \, dx = \int_{\Omega} f(t, \cdot) \varphi \, dx$$

for all $t \in [0, T]$ and all $\varphi \in L^\infty(\Omega)$.

Uniqueness

Theorem (Uniqueness)

Let Θ be such that φ_Θ is **concave** and **Osgood**. There is **at most one** (Lagrangian) weak solution (ω, v) of (E) such that

$$\omega \in L_{loc}^\infty([0, +\infty); L^1(\Omega) \cap Y_{ul}^\Theta(\Omega)) \quad v \in L_{loc}^\infty([0, +\infty); C_b^{0, \varphi_\Theta}(\Omega; \mathbb{R}^2)),$$

starting from $\omega_0 \in L^1(\Omega) \cap Y_{ul}^\Theta(\Omega)$, $v_0 = K\omega_0$.

Remark: our uniqueness result

- recovers (and actually improves) Yudovich's uniqueness theorem;
- is proved in a Lagrangian way, we do not use the energy method;
- does not rely on the specific structure of the Biot-Savart kernel.

Careful: Osgood velocity \Rightarrow **any** weak solution is Lagrangian, but this is delicate!

- Ambrosio-Bernard (2008)
- Caravenna-Crippa (2021)
- Clop-Jylhä-Mateu-Orotobig (2019)

Proof of uniqueness 1/4

Assume (ω^1, v^1) and (ω^2, v^2) are two **Lagrangian** solutions with same initial datum.

We can thus write $\omega^i = X^i(t, \cdot) \# \omega_0$ where X^i is the flow associated to v^i , $i = 1, 2$.

Fix $T > 0$ and consider $t \in [0, T]$. We start with the usual splitting

$$\begin{aligned} |X^1 - X^2| &\leq \int_0^t |v^1(s, X^1) - v^2(s, X^2)| ds \\ &\leq \int_0^t |v^1(s, X^1) - v^1(s, X^2)| ds + \int_0^t |v^1(s, X^2) - v^2(s, X^2)| ds. \end{aligned}$$

The first term is easy, we can use the **φ_Θ -continuity** and obtain

$$|v^1(s, X^1) - v^1(s, X^2)| \lesssim \varphi_\Theta(|X^1 - X^2|),$$

with implicit constant depending on $\|\omega^1\|_{L^\infty([0, T]; L^1 \cap Y_{ul}^\Theta)}$.

Proof of uniqueness 2/4

The second term is delicate. We use $v = K\omega$ and the **push-forward** to get

$$\begin{aligned} |v^1(s, X^2) - v^2(s, X^2)| &= |(K\omega^1)(s, X^2) - (K\omega^2)(s, X^2)| \\ &= \left| \int_{\Omega} k(X^2, y) \omega^1(s, y) dy - \int_{\Omega} k(X^2, y) \omega^2(s, y) dy \right| \\ &= \left| \int_{\Omega} k(X^2, X^1(s, y)) \omega_0(y) dy - \int_{\Omega} k(X^2, X^2(s, y)) \omega_0(y) dy \right| \\ &\leq \int_{\Omega} |k(X^2, X^1(s, y)) - k(X^2, X^2(s, y))| |\omega_0(y)| dy. \end{aligned}$$

We combine the two estimates and obtain

$$\begin{aligned} |X^1 - X^2| &\leq \int_0^t \varphi_{\Theta}(|X^1 - X^2|) dt \\ &\quad + \int_0^t \int_{\Omega} |k(X^2, X^1(s, y)) - k(X^2, X^2(s, y))| |\omega_0(y)| dy dt. \end{aligned}$$

Now choose the **finite** measure $\mu = \bar{\omega} \mathcal{L}^2$, with $\bar{\omega} = |\omega_0| + \eta$ and $0 < \eta \in L^1 \cap L^\infty$.

Proof of uniqueness 3/4

We integrate with respect to μ . By Tonelli Theorem, we can estimate

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} |k(X^2(s, x), X^1(s, y)) - k(X^2(s, x), X^2(s, y))| |\omega_0(y)| dy d\mu(x) \\ &= \int_{\Omega} |\omega_0(y)| \int_{\Omega} |k(X^2(s, x), X^1(s, y)) - k(X^2(s, x), X^2(s, y))| d\mu(x) dy \\ &= \int_{\Omega} |\omega_0(y)| \int_{\Omega} |k(x, X^1(s, y)) - k(x, X^2(s, y))| X^2(s, \cdot) \# \bar{\omega}(x) dx dy \\ &\stackrel{(!)}{\lesssim} \int_{\Omega} |\omega_0(y)| \varphi_{\Theta}(|X^1(s, y) - X^2(s, y)|) dy \\ &\leq \int_{\Omega} \varphi_{\Theta}(|X^1(s, y) - X^2(s, y)|) d\mu(y). \end{aligned}$$

Inequality (!) follows from the same computations for the φ_{Θ} -continuity of velocity.

The implicit constant depends on $\|\bar{\omega}\|_{L^{\infty}([0, T]; L^1 \cap Y_{\mu}^{\Theta})}$. But $\bar{\omega} = |\omega_0| + \eta$, so we can choose $\eta \in L^1 \cap L^{\infty}$ to let the constant depend on $\|\omega_0\|_{L^{\infty}([0, T]; L^1 \cap Y_{\mu}^{\Theta})}$ only!

Proof of uniqueness 4/4

In conclusion, we get

$$\int_{\Omega} |X^1 - X^2| d\mu \lesssim \int_0^t \int_{\Omega} \varphi_{\Theta}(|X^1 - X^2|) d\mu dt.$$

But φ_{Θ} is **concave** and **Osgood**, so that

$$\int_{\Omega} \varphi_{\Theta}(|X^1 - X^2|) d\mu \stackrel{\text{Young}}{\leq} \varphi_{\Theta} \left(\int_{\Omega} |X^1 - X^2| d\mu \right)$$

and thus

$$\xi(t) \leq \int_0^t \varphi_{\Theta}(\xi(s)) dt, \quad \xi(s) = \int_{\Omega} |X^1(s, \cdot) - X^2(s, \cdot)| d\mu,$$

imply that $X^1 = X^2$ for all $t \in [0, T]$, which means $\omega^1 = \omega^2$ and so $v^1 = v^2$.

Project 1: apply this elementary approach to **Vlasov-Poisson system**

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 \\ E = K \varrho \\ \varrho = \int f \, dv, \end{cases}$$

in collaboration with G. Crippa, T. Dolmaire and C. Saffirio.

Project 2: remove L^1 assumption in **existence**, constructing weak solutions in Y_{ul}^Θ for suitable Θ , in collaboration with G. Ciampa and G. Crippa.

Other ideas: more general functional spaces? lake equations?

Thank you for your attention!

G. Crippa and G. Stefani, "An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces" (2021), submitted, available at [arXiv:2110.15648](https://arxiv.org/abs/2110.15648).

Slides available upon request (giorgio.stefani@unibas.ch) or on giorgiostefani.weebly.com.