A distributional approach to fractional spaces and fractional variation

Giorgio Stefani

Universität Basel

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[1] G. E. Comi and G. Stefani, "A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up", J. Funct. Anal. 277 (2019), no. 10, 3373-3435.

[2] G. E. Comi and G. Stefani, "A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics I" (2019), submitted, available at arXiv:1910.13419.

[3] E. Bruè, M. Calzi, G. E. Comi and G. Stefani, "A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics II" (2020), submitted, available at arXiv:2011.03928.

[4] G. E. Comi, D. Spector and G. Stefani, in preparation.

Fractional derivatives: three famous examples

Around 1675 Newton and Leibniz discovered Calculus. Somewhat surprisingly, the first appearance of the concept of a fractional derivative is found in a letter written to De l'Hôpital by Leibniz in 1695!

Let us recall the three most famous fractional derivatives:

Leibniz-Lacroix (1819): $$\frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Riemann-Liouville (1832-1847): $${}^{RL}D^\alpha_a f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha}\, d\tau$$

Caputo (1967): $${}^{C}D^\alpha_a f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau.$$

Some observations:

- they are defined just for functions of one variable;
- only Caputo's derivative kills constants;
- Caputo's derivative requires f to be differentiable!

Question: What about fractional gradient? Can we just take $\left(D^{\alpha,1}, \ldots, D^{\alpha,n}\right)$?

Be careful: the "coordinate approach" gives an operator not invariant by rotations!

Silhavy's approach: invariance properties

Recently, Silhavy proposed that a "good" fractional operator should satisfy:

- invariance with respect to translations and rotations;
- α-homogeneity for some $\alpha \in (0,1)$;
- mild continuity on suitable test space, e.g. C_c^∞ or Schwartz's space $\mathscr{S}$.

Idea behind: fractional operators should have a physical meaning!

For $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we consider

$$\nabla^\alpha f(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(y-x)}{|y-x|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n,$$

and

$$\mathrm{div}^\alpha \varphi(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n.$$

Theorem (Silhavy, 2020)

∇^α and div^α are determined (up to mult. const.) by the three requirements above.

A little bit of literature on ∇^α

The operator fractional Riesz gradient $\nabla^\alpha \equiv \nabla I_{1-\alpha}$ has a long story:

- 1959, Horvath (earliest reference up to knowledge);
- 1961, implicitly mentioned in one paper by Nikol'ski-Sobolev;
- 1971, non-local continuum mechanics by Edelen-Green-Laws;
- 2011-13, non-local porous medium equation by Caffarelli-Soria-Vazquez;
- 2015, non-local porous medium equation by Biler-Imbert-Karch;
- after 2015, fractional PDE theory and "geometric" inequalities by Shieh-Spector, Ponce-Spector, Schikorra-Spector-Van Schaftingen;
- 2020, distributional approach by Silhavy (introducing $\mathrm{div}^\alpha \equiv \mathrm{div} I_{1-\alpha}$).

Duality, fractional Laplacian and Riesz transform

The operators ∇^α and div^α are dual, in the sense that

$$\int_{\mathbb{R}^n} f \, \mathrm{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

The operators ∇^α and div^β satisfy $-\mathrm{div}^\beta \nabla^\alpha = (-\Delta)^{\frac{\alpha+\beta}{2}}$.

If we let

$$I_\alpha u(x) := \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}} \, dy$$

be the fractional Riesz potential of $u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^m)$, then

$$\nabla^\alpha f = I_{1-\alpha} \nabla f, \qquad \mathrm{div}^\alpha \varphi = I_{1-\alpha} \, \mathrm{div} \varphi.$$

Integrability: $\nabla^\alpha f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\mathrm{div}^\alpha \varphi \in L^1(\mathbb{R}^n; \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

Relax: ∇^α and div^α are well posed also for Lip_c-regular test functions.

Leibniz's rules for ∇^α and $\operatorname{div}^\alpha$

For any $f, g \in C_c^\infty(\mathbb{R}^n)$, we have it holds

$$\nabla^\alpha(fg) = f\nabla^\alpha g + g\nabla^\alpha f + \nabla^\alpha_{\mathrm{NL}}(f,g),$$

where

$$\nabla^\alpha_{\mathrm{NL}}(f,g)(x) := \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(g(y)-g(x))(y-x)}{|y-x|^{n+\alpha+1}}\,dy, \quad x \in \mathbb{R}^n.$$

For any $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we also have it holds

$$\operatorname{div}^\alpha(f\varphi) = f\operatorname{div}^\alpha\varphi + \varphi\cdot\nabla^\alpha f + \operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi),$$

where

$$\operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi)(x) := \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy, \quad x \in \mathbb{R}^n.$$

Although ∇^α and $\operatorname{div}^\alpha$ have a strong non-local behaviour, they commute with convolutions (by linearity). In addition, Leibniz's rules allow for approximation by cut-off functions (careful control on non-local terms).

A fractional version of the Fundamental Theorem of Calculus

Let $\alpha \in (0,1)$. If $f \in C_c^\infty(\mathbb{R}^n)$, then

$$f(y) - f(x) = \mu_{n,-\alpha} \int_{\mathbb{R}^n} \left(\frac{z-x}{|z-x|^{n+1-\alpha}} - \frac{z-y}{|z-y|^{n+1-\alpha}} \right) \cdot \nabla^\alpha f(z)\, dz$$

for any $x, y \in \mathbb{R}^n$.

Some good news:

- we get L^1-control on translations;
- we get L^1-control on smoothed-by-convolution functions;
- we get compactness for sequences with uniformly bounded RHS.

Some bad news:

- left-hand integral is on the whole space (non-locality!);
- we cannot get local Poincaré inequality;
- we cannot get relative fractional isoperimetric inequality.

Fractional variation and the space $BV^\alpha(\mathbb{R}^n)$

We define

$$BV^\alpha(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty\},$$

where

$$|D^\alpha f|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\}.$$

In perfect analogy with the classical BV framework:

- $BV^\alpha(\mathbb{R}^n)$ is a Banach space and its norm is l.s.c. w.r.t. L^1-convergence;
- $C^\infty(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ are dense subspaces of $BV^\alpha(\mathbb{R}^n)$;
- given $f \in L^1(\mathbb{R}^n)$, $f \in BV^\alpha(\mathbb{R}^n) \iff \exists D^\alpha f \in \mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$ such that

  $$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f$$

  for any $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$;
- unif. bounded seq. in $BV^\alpha(\mathbb{R}^n)$ admit limit points in $L^1(\mathbb{R}^n)$ w.r.t. L^1_{loc}-conv.;
- for $n \ge 2$ we have $BV^\alpha(\mathbb{R}^n) \subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ (Gagliardo-Nirenberg-Sobolev).

Fractional distributional Sobolev spaces and Bessel potential spaces

For $p \in [1, +\infty]$, we define the distributional fractional Sobolev space

$$S^{\alpha,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : \exists\, \nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n)\}.$$

Here $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ is the weak fractional gradient of $f \in L^p(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n).$$

We naturally have $S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ with

$$f \in S^{\alpha,1}(\mathbb{R}^n) \iff |D^\alpha f| \ll \mathscr{L}^n,\ D^\alpha f = \nabla^\alpha f\, \mathscr{L}^n.$$

We are also able to prove that $BV^\alpha(\mathbb{R}^n) \setminus S^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$.

Theorem (Bruè-Calzi-Comi-S., 2020)

If $p \in (1, +\infty)$, then $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$, where

$$L^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : (I - \Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n)\}$$

is the Bessel potential space.

Fractional Sobolev spaces and fractional operators

For $p \in [1, +\infty)$ and $\alpha \in (0,1)$, we let

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+p\alpha}} \, dx \, dy < +\infty \right\}.$$

The fractional perimeter in an open set $\Omega \subset \mathbb{R}^n$ of a measurable set $E \subset \mathbb{R}^n$ is

$$P_\alpha(E;\Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}} \, dx \, dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}} \, dx \, dy.$$

If $\Omega = \mathbb{R}^n$, then $P_\alpha(E;\mathbb{R}^n) = P_\alpha(E) = [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)}$.

Notice that we have the extension $\nabla^\alpha \colon W^{\alpha,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n;\mathbb{R}^n)$, since

$$\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \mu_{n,\alpha} [f]_{W^{\alpha,1}(\mathbb{R}^n)} \text{ for all } f \in W^{\alpha,1}(\mathbb{R}^n).$$

We thus have $W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n)$ with $f \in W^{\alpha,1}(\mathbb{R}^n) \Rightarrow D^\alpha f = \nabla^\alpha f \mathscr{L}^n$.

Since $W^{\alpha,1}(\mathbb{R}^n)$ is closed w.r.t. pointwise convergence, $S^{\alpha,1}(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$.

Remarkably, if $0 < \beta < \alpha < 1$ then $BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$.

Sets of finite fractional Caccioppoli α-perimeter

In perfect analogy with the standard BV setting, we give the following definition.

Let $\alpha \in (0,1)$ and $E \subset \mathbb{R}^n$ be a measurable set. For any open set $\Omega \subset \mathbb{R}^n$, we let

$$|D^\alpha \chi_E|(\Omega) = \sup\left\{ \int_E \mathrm{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\}$$

be the fractional Caccioppoli α-perimeter of E in Ω. If $|D^\alpha \chi_E|(\Omega) < +\infty$, then E has finite fractional Caccioppoli α-perimeter in Ω.

Note that $E \subset \mathbb{R}^n$ has finite fractional Caccioppoli α-perimeter in Ω if and only if $D^\alpha \chi_E \in \mathcal{M}(\Omega; \mathbb{R}^n)$ and

$$\int_E \mathrm{div}^\alpha \varphi \, dx = - \int_\Omega \varphi \cdot dD^\alpha \chi_E$$

for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$.

Question: can we define a fractional version of De Giorgi's reduce boundary?

Fractional reduced boundary

It is now natural to give the following definition.

Let $E \subset \mathbb{R}^n$ be a set with finite fractional Caccioppoli α-perimeter in Ω. A point $x \in \Omega$ belongs to the fractional reduced boundary of E (inside Ω) if

$$x \in \operatorname{supp}(D^\alpha \chi_E) \qquad \text{and} \qquad \exists \lim_{r \to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} \in \mathbb{S}^{n-1}.$$

We thus let $\mathscr{F}^\alpha E$ be the fractional reduced boundary of E and define

$$\nu_E^\alpha \colon \Omega \cap \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}, \qquad \nu_E^\alpha(x) := \lim_{r \to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))}, \quad x \in \Omega \cap \mathscr{F}^\alpha E,$$

the inner unit fractional normal to E (inside Ω).

We thus have the following Gauss-Green formula

$$\int_E \operatorname{div}^\alpha \varphi \, dx = - \int_{\Omega \cap \mathscr{F}^\alpha E} \varphi \cdot \nu_E^\alpha \, d|D^\alpha \chi_E|.$$

for all $\varphi \in \operatorname{Lip}_c(\Omega; \mathbb{R}^n)$.

Sets of finite fractional perimeter

If $E \subset \mathbb{R}^n$ satisfies $P_\alpha(E;\Omega) < +\infty$, then

$$|D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E;\Omega)$$

and

$$D^\alpha \chi_E = \nu_E^\alpha \, |D^\alpha \chi_E| = \nabla^\alpha \chi_E \, \mathscr{L}^n.$$

Moreover, if $\chi_E \in BV(\mathbb{R}^n)$, then

$$\nabla^\alpha \chi_E(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{\nu_E(y)}{|y-x|^{n+\alpha-1}} \, d|D\chi_E|(y)$$

for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Be careful! We have

$$P_\alpha(E;\Omega) < +\infty \Rightarrow \mathscr{L}^n(\Omega \cap \mathscr{F}^\alpha E) > 0$$

even including the case $\chi_E \in BV(\mathbb{R}^n)$. In other words, the non-local operator ∇^α produces a diffuse fractional boundary in the $W^{\alpha,1}$ regime (since $W^{\alpha,1} \subset S^{\alpha,1}$).

Example: $E = (a,b) \subset \mathbb{R} \Rightarrow \mathscr{F}^\alpha E = \mathbb{R} \setminus \left\{\frac{a+b}{2}\right\}$!

Two examples: balls and halfspaces

Example 1. For $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$, we have

$$\nabla^\alpha \chi_{B_1}(x) = -\frac{\mu_{n,\alpha}}{n+\alpha-1} g_{n,\alpha}(|x|) \frac{x}{|x|},$$

where

$$g_{n,\alpha}(t) := \int_{\partial B_1} \frac{y_1}{|t\mathsf{e}_1 - y|^{n+\alpha-1}} \, d\mathscr{H}^{n-1}(y) > 0, \quad \text{for any } t \geq 0,$$

which means $\nu_{B_1}^\alpha(x) = -x/|x|$ for any $x \neq 0$ and $\mathscr{F}^\alpha B_1 = \mathbb{R}^n \setminus \{0\}$.

Example 2. For the halfspace $H_\nu^+ = \{y \cdot \nu \geq 0\}$, if $x \cdot \nu \neq 0$ then

$$\nabla^\alpha \chi_{H_\nu^+}(x) = \frac{2^{\alpha-1}\Gamma\left(\frac{\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1-\alpha}{2}\right)} \frac{1}{|x \cdot \nu|^\alpha} \nu.$$

In particular, $\mathscr{F}^\alpha H_\nu^+ = \mathbb{R}^n$ and $\nu_{H_\nu^+}^\alpha \equiv \nu$.

Density estimates

Thanks to the invariance properties, we get

$$D^\alpha \chi_{\frac{E-x}{r}} = \frac{1}{r^{n-\alpha}} (I_{x,r})_\# D^\alpha \chi_E,$$

where $I_{x,r}(y) = (y-x)/r$ for $x, y \in \mathbb{R}^n$ and $r > 0$.

Theorem (Comi-S., 2019)

If $E \subset \mathbb{R}^n$ has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, then for any $x \in \mathscr{F}^\alpha E$ there exists $r_x > 0$ such that

$$|D^\alpha \chi_E|(B_r(x)) \le A_{n,\alpha} r^{n-\alpha}, \quad |D^\alpha \chi_{E \cap B_r(x)}|(\mathbb{R}^n) \le B_{n,\alpha} r^{n-\alpha}$$

for all $r \in (0, r_x)$.

By a standard covering argument, we thus get that

$$|D^\alpha \chi_E| \le C_{n,\alpha} \mathscr{H}^{n-\alpha} \llcorner \mathscr{F}^\alpha E$$

and therefore

$$\dim_{\mathscr{H}}(\mathscr{F}^\alpha E) \ge n - \alpha.$$

Existence of blow-ups and coarea inequality

Let Tan(E, x) be the set of all tangent sets of E at x, i.e. the set of all limit points in $L^1_{\text{loc}}(\mathbb{R}^n)$-topology of the family $\left\{\frac{E-x}{r} : r > 0\right\}$ as $r \to 0^+$.

Theorem (Comi-S., 2019)

- If E has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, then Tan$(E, x) \neq \varnothing$ for all $x \in \mathscr{F}^\alpha E$.
- Moreover, if $F \in$ Tan(E, x), then F has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ and $\nu_F^\alpha(y) = \nu_E^\alpha(x)$ for $|D^\alpha \chi_F|$-a.e. $y \in \mathscr{F}^\alpha F$.

What is missing: density estimates from below and we need coarea fromula.

Theorem (Comi-S., 2019)

If $f \in BV^\alpha(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|(\mathbb{R}^n)\, dt < +\infty$, then

$$D^\alpha f = \int_{\mathbb{R}} D^\alpha \chi_{\{f>t\}}\, dt, \qquad |D^\alpha f| \leq \int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|\, dt.$$

Bad news: there exist $f \in BV^\alpha(\mathbb{R}^n)$ such that $\int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|(\mathbb{R}^n)\, dt = +\infty$!

Asymptotics as $\alpha \to 1^-$

Theorem (Bourgain-Brezis-Mironescu & Davila, 2002)

Let $p \in [1, +\infty)$. If $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} (1-\alpha)[f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = c_{n,p} \|\nabla f\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^n)}.$$

If $f \in BV(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} (1-\alpha)[f]_{W^{\alpha,1}(\mathbb{R}^n)} = c_{n,1} |Df|(\mathbb{R}^n).$$

Now it is important to observe that

$$\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \sim \frac{1-\alpha}{\omega_n} \quad \text{as } \alpha \to 1^-.$$

Theorem (Comi-S., 2019)

Let $p \in [1, +\infty)$. If $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} \|\nabla^\alpha f - \nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0.$$

If $f \in BV(\mathbb{R}^n)$, then $D^\alpha f \rightharpoonup Df$ and $|D^\alpha f| \rightharpoonup |Df|$ as $\alpha \to 1^-$ and moreover

$$\lim_{\alpha \to 1^-} |D^\alpha f|(\mathbb{R}^n) = |Df|(\mathbb{R}^n).$$

Γ-convergence as $\alpha \to 1^-$

Theorem (Ambrosio-De Philippis-Martinazzi, 2011)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$\Gamma(L^1_{\text{loc}})\text{-}\lim_{\alpha \to 1^-} (1-\alpha) P_\alpha(E;\Omega) = 2\omega_{n-1} P(E;\Omega)$$

for every measurable set $E \subset \mathbb{R}^n$.

Theorem (Comi-S., 2019)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$\Gamma(L^1_{\text{loc}})\text{-}\lim_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) = P(E;\Omega)$$

for every measurable set $E \subset \mathbb{R}^n$.

Theorem (Comi-S., 2019)

Let $\Omega \subset \mathbb{R}^n$ be an open set such that Ω is bounded with Lipschitz boundary or $\Omega = \mathbb{R}^n$. Then

$$\Gamma(L^1)\text{-}\lim_{\alpha \to 1^-} |D^\alpha f|(\Omega) = |Df|(\Omega)$$

for every $f \in BV(\mathbb{R}^n)$.

The space $BV^0(\mathbb{R}^n)$ and the Hardy space

Imitating what we did before, we define

$$BV^0(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |D^0 f|(\mathbb{R}^n) < +\infty\},$$

where

$$|D^0 f|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \,\mathrm{div}^0\varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\}.$$

Here div^0 is the dual operator of $\nabla^0 = I_1\nabla = R$, the Riesz transform

$$Rf(x) = \mu_{n,0}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(y-x)}{|y-x|^{n+1}}\, dy, \quad x \in \mathbb{R}^n,$$

(in the principal value sense).

Theorem (Bruè-Calzi-Comi-S., 2020)

We have $BV^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, where

$$H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : Rf \in L^1(\mathbb{R}^n;\mathbb{R}^n)\}$$

is the (real) Hardy space, with $D^0 f = Rf\, \mathscr{L}^n$ as measures for $f \in BV^0(\mathbb{R}^n)$.

Asymptotics as $\alpha \to 0^+$

Theorem (Maz'ya-Shaposhnikova, 2002)

Let $p \in [1, +\infty)$. If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \alpha \, [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = c_{n,p} \|f\|^p_{L^p(\mathbb{R}^n)}.$$

Now it is important to observe that $\mu_{n,\alpha} \to \mu_{n,0}$ as $\alpha \to 0^+$ (no rescaling!).

Theorem (Bruè-Calzi-Comi-S., 2020)

Let $p \in (1, +\infty)$. If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \|\nabla^\alpha f - Rf\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)} = 0.$$

If $f \in H^1(\mathbb{R}^n) \cap \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then (actually, in H^1 norm)

$$\lim_{\alpha \to 0^+} \|\nabla^\alpha f - Rf\|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} = 0.$$

If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \alpha \int_{\mathbb{R}^n} |\nabla^\alpha f(x)| \, dx = n\omega_n \mu_{n,0} \left| \int_{\mathbb{R}^n} f(x) \, dx \right|.$$

Fractional interpolation inequalities

We prove $\nabla^\alpha \to R$ strongly as $\alpha \to 0^+$ via fractional interpolation inequalities.

Theorem (Bruè-Calzi-Comi-S., 2020)

Let $\alpha \in (0,1]$. There exists $c_{n,\alpha} > 0$ such that

$$|D^\beta f|(\mathbb{R}^n) \le c_{n,\alpha} \|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} |D^\alpha f|(\mathbb{R}^n)^{\frac{\beta}{\alpha}}$$

for all $\beta \in [0,\alpha]$ and all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$.

Theorem (Bruè-Calzi-Comi-S., 2020)

Let $p \in (1,+\infty)$. There exists $c_{n,p} > 0$ such that

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,p} \|\nabla^\gamma f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$. There exists $c_n > 0$ such that

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)} \le c_n \|\nabla^\gamma f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in HS^{\alpha,1}(\mathbb{R}^n)$ (i.e., $f \in H^1$ and $\nabla^\alpha f \in H^1$).

$BV^{\alpha,p}(\mathbb{R}^n)$ and the properties of the fractional variation

Given $p \in [1, +\infty]$, we define

$$BV^{\alpha,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty\},$$

where, as before,

$$|D^\alpha f|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\}.$$

The space $BV^{\alpha,p}(\mathbb{R}^n)$ is similar to the original space $BV^\alpha(\mathbb{R}^n) = BV^{\alpha,1}(\mathbb{R}^n)$, but the integrability exponent p plays an interesting role.

Theorem (Comi-Spector-S., in preparation)

Let $\alpha \in (0,1)$, $p \in [1, +\infty]$ and assume that $f \in BV^{\alpha,p}(\mathbb{R}^n)$.

- (subcritical case) If $p \in \left[1, \frac{n}{1-\alpha}\right)$, then $|D^\alpha f| \ll \mathscr{H}^{n-1}$.
- (supercritical case) If $p \in \left[\frac{n}{1-\alpha}, +\infty\right]$, then $|D^\alpha f| \ll \mathscr{H}^{\frac{n}{q}-\alpha}$, with $\frac{1}{p} + \frac{1}{q} = 1$.

Remark: $I_{1-\alpha} : BV^{\alpha,p}(\mathbb{R}^n) \to BV^{1,\frac{np}{n-(1-\alpha)p}}(\mathbb{R}^n)$ is continuous for $p \in \left(1, \frac{n}{1-\alpha}\right)$.

Open problems and research directions [1/2]

About sets and perimeter. With the distributional approach to fractional variation, many research directions are interesting.

▹ We proved that $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ strictly only at the level of functions. Is there $\chi_E \in BV^\alpha(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n)$ for some measurable set $E \subset \mathbb{R}^n$?

▹ We know that blow-ups exist and have constant fractional normal. Can we characterise them more precisely? Are they unique (in some cases)?

▹ Minimal surfaces for P_α are widely studied. What about minimal surfaces for the fractional variation? Can we perform calibrations?

▹ Isoperimetric sets for P_α are balls (also in a quantitative sense). Are balls isoperimetric sets for the fractional variation?

Open problems and research directions [2/2]

About interpolation. Fractional interpolation inequalities may be derived from real/complex Interpolation Theory.

▷ What is the real interpolation space between $BV(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$? Note that

$$(H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\alpha,p} \subset (L^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\alpha,p} = B^\alpha_{1,p}(\mathbb{R}^n) \supset W^{\alpha,1}(\mathbb{R}^n)$$

and that $B^\alpha_{1,1}(\mathbb{R}^n) = W^{\alpha,1}(\mathbb{R}^n)$.

▷ What is the complex interp. space between $BV(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$? Note that

$$\begin{array}{ccccc}
(H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\alpha,1} & \subset & (H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{[\alpha]} & \subset & (H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\alpha,\infty} \\
\cap & & & & \cap \\
W^{\alpha,1}(\mathbb{R}^n) & \subsetneq & BV^\alpha(\mathbb{R}^n) & \subsetneq & B^\alpha_{1,\infty}(\mathbb{R}^n)
\end{array}$$

(with the second inclusion strict for all $n \geq 2$).

About the general theory. What is the "right" definition of BV^α on a general open set $\Omega \subset \mathbb{R}^n$? We would like to keep integration by parts, but what is the role of $\partial\Omega$?

Thank you for your attention!