Bakry-Émery curvature condition and entropic inequalities on metric-measure groups

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In $\mathbb{R}^N$ the solution of heat equation

$$\begin{cases} \partial_t f_t = \Delta f_t & \text{on } \mathbb{R}^N \times (0, +\infty) \\ f_0 = f & \text{on } \mathbb{R}^N \end{cases}$$

is given by convolution as $\mathsf{P}_t f = \mathsf{p}_t * f$, where

$$\mathsf{p}_t(x) = \frac{1}{(4\pi t)^{N/2}}\, e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^N,$$

is the heat kernel.

Hence we have $\nabla \mathsf{P}_t f = \mathsf{p}_t * (\nabla f) = \mathsf{P}_t \nabla f$, so that

$$\Gamma(\mathsf{P}_t f) = |\nabla \mathsf{P}_t f|^2 = |\mathsf{P}_t \nabla f|^2 \le \mathsf{P}_t(|\nabla f|^2) = \mathsf{P}_t \Gamma(f)$$

by Jensen inequality.

Bochner formula and Bakry-Émery CD inequality

Let $(\mathbb{M}, \mathrm{g})$ be a (complete and connected) N-dimensional smooth Riemannian manifold with Laplace-Beltrami operator Δ. The Bochner formula states that

$$\frac{1}{2}\,\Delta|\nabla f|^2_{\mathrm{g}} = \langle \nabla \Delta f, \nabla f \rangle_{\mathrm{g}} + ||\mathrm{Hess} f||_2^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

Define

$$\Gamma(f,g) = \langle \nabla f, \nabla g \rangle_{\mathrm{g}}, \quad \Gamma_2(f,g) = \frac{1}{2}\big(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g)\big).$$

Then the Bochner formula gives

$$\frac{1}{2}\,\Delta\Gamma(f) = \Gamma(\Delta f, f) + ||\mathrm{Hess} f||_2^2 + \mathrm{Ric}(\nabla f, \nabla f),$$

so that

$$\Gamma_2(f) = ||\mathrm{Hess} f||_2^2 + \mathrm{Ric}(\nabla f, \nabla f),$$

where $\Gamma(f) = \Gamma(f,f)$ and $\Gamma_2(f) = \Gamma_2(f,f)$ for simplicity.

Using Cauchy-Schwartz inequality, we can estimate

$$||\mathrm{Hess} f||_2^2 \geq \frac{1}{N}\,(\Delta f)^2,$$

and so we get Bakry-Émery curvature-dimension inequality $\mathsf{CD}(K,N)$

$$\dim \mathbb{M} \leq N,\ \mathrm{Ric} \geq K \iff \Gamma_2(f) \geq \frac{1}{N}\,(\Delta f)^2 + K\,\Gamma(f).$$

Assume that $(\mathbb{M}, \mathsf{g})$ satisfies Ric $\geq K$ (take $N = \infty$).

Define $\varphi(s) = P_s\Gamma(P_{t-s}f)$ for $s \in [0,t]$. Then

$$\varphi'(s) = 2P_s\Gamma_2(P_{t-s}f) \overset{\mathsf{CD}(K,\infty)}{\geq} 2KP_s\Gamma(P_{t-s}f) = 2K\varphi(s)$$

and thus, by Grönwall inequality,

$$\Gamma(\mathsf{P}_t f) \leq e^{-2Kt}\,\mathsf{P}_t\Gamma(f), \qquad \text{(BE)}$$

the Bakry-Émery pointwise gradient estimate for the heat flow. If $\mathbb{M} = \mathbb{R}^N$, then $K = 0$ and we recover the Euclidean case.

Differentiating (BE) at $t = 0$ we get $\mathsf{CD}(K, \infty)$.

This argument works also for the case $N < \infty$ [Wang, 2011].

Warm-up in $\mathbb{H}^1$

On the manifold $\mathbb{R}^3$ consider the non-commutative group operation

$$p \bullet q = (x, y, z) \bullet (x', y', z') = \left(x + x', y + y', z + z' + \tfrac{1}{2}(xy' - yx')\right).$$

The resulting Lie group $(\mathbb{R}^3, \bullet) \equiv \mathbb{H}^1$ is the (first) Heisenberg group.

There is a family of dilations: $\delta_\lambda(p) = (\lambda x, \lambda y, \lambda^2 z)$.

The Haar measure is the Lebesgue measure $\mathscr{L}^3 = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

The tangent space is spanned by

$$X = \partial_x - \tfrac{y}{2}\partial_z, \quad Y = \partial_y + \tfrac{x}{2}\partial_z, \quad Z = [X, Y] = \partial_z.$$

We use the horizontal generators X, Y to define

$$\mathsf{d}_{CC}(p, q) = \inf\left\{\int_0^1 \|\dot\gamma_s\|_{\mathbb{H}^1}\,\mathrm{d}s : \gamma_0 = p,\ \gamma_1 = q,\ \dot\gamma_s \in \operatorname{span}\{X_{\gamma_s}, Y_{\gamma_s}\}\right\}.$$

The function d_{CC} is the Carnot-Carathéodory (CC) distance [Chow-Rashevskii].

We work with the (not-that-bad) metric-measure space $(\mathbb{H}^1, \mathsf{d}_{CC}, \mathscr{L}^3)$.

What about a BE gradient estimate in $\mathbb{H}^1$?

The canonical sub-Laplacian in $\mathbb{H}^1$ is $\Delta_{\mathbb{H}^1} = X^2 + Y^2$, which is not elliptic. But $\Delta_{\mathbb{H}^1}$ is hypoelliptic, so the fundamental solution p_t of $\partial_t - \Delta_{\mathbb{H}^1}$ is smooth [Hörmander]. In $\mathbb{H}^1$ the solution of the (sub-elliptic) heat equation

$$\begin{cases} \partial_t f_t = \Delta_{\mathbb{H}^1} f_t & \text{on } \mathbb{R}^3 \times (0, +\infty) \\ f_0 = f & \text{on } \mathbb{R}^3 \end{cases}$$

is thus given by convolution as

$$\mathsf{P}_t f(p) = \mathsf{p}_t \star f(p) = \int_{\mathbb{R}^3} \mathsf{p}_t(q^{-1}p)\, f(q)\, \mathrm{d}q = \int_{\mathbb{R}^3} \mathsf{p}_t(q)\, f(pq^{-1})\, \mathrm{d}q$$

The horizontal gradient $\nabla_{\mathbb{H}^1} = (X, Y)$ is only left-invariant, so

$$\nabla_{\mathbb{H}^1} \mathsf{P}_t f = \nabla_{\mathbb{H}^1} (\mathsf{p}_t \star f) = (\nabla_{\mathbb{H}^1} \mathsf{p}_t) \star f \neq \mathsf{p}_t \star (\nabla_{\mathbb{H}^1} f) = \mathsf{P}_t(\nabla_{\mathbb{H}^1} f).$$

Theorem (Driver - Melcher, 2005)

There exists $C_{\mathbb{H}^1} > 1$ such that $\Gamma^{\mathbb{H}^1}(\mathsf{P}_t f) \leq C_{\mathbb{H}^1}^2 \mathsf{P}_t \Gamma^{\mathbb{H}^1}(f)$.

This is a weak BE gradient estimate in $\mathbb{H}^1$: no differentiation at time $t = 0$!

Let (X, d) be a Polish (geodesic) metric space. We endow the set

$$\mathscr{P}_2(X) = \left\{ \mu \in \mathscr{P}(X) : \int_X \mathsf{d}(x, x_0)^2 \,\mathrm{d}\mu(x) < +\infty, \; x_0 \in X \right\}$$

with the Wasserstein distance

$$W_2^2(\mu, \nu) = \inf \left\{ \int_{X \times X} \mathsf{d}^2(x, y) \,\mathrm{d}\pi : \pi(x, y) \in \mathsf{Plan}(\mu, \nu) \right\},$$

where

$$\mathsf{Plan}(\mu, \nu) = \{ \pi \in \mathscr{P}(X \times X) : (p_1)_\# \pi = \mu, \; (p_2)_\# \pi = \nu \}.$$

Fact: $(\mathscr{P}_2(X), W_2)$ is a Polish (geodesic) metric space.

By Kantorovich duality formula [Fenchel-Rockafellar duality principle]

$$\frac{1}{2} W_2^2(\mu, \nu) = \sup \left\{ \int_X Q_1 \varphi \,\mathrm{d}\mu - \int_X \varphi \,\mathrm{d}\nu : \varphi \in \mathrm{Lip}(X) \text{ with bounded support} \right\}$$

where

$$Q_s \varphi(x) = \inf_{y \in X} \varphi(y) + \frac{d^2(y, x)}{2s}$$

for $s > 0$ with $Q_0 \varphi = \varphi$ is the Hopf-Lax semigroup.

Kuwada duality

Assume $(X, \mathsf{d}) = (\mathbb{M}, \mathsf{g})$. We have another equivalent characterization of Ric $\geq K$.

Theorem (von Renesse - Sturm, 2005)

$$\Gamma(\mathsf{P}_t f) \leq e^{-2Kt}\, \mathsf{P}_t \Gamma(f) \iff W_2(\mathsf{P}_t\mu, \mathsf{P}_t\nu) \leq e^{-Kt}\, W_2(\mu,\nu)$$

A similar result is available for $(X, \mathsf{d}) = (\mathbb{H}^1, \mathsf{d}_{CC})$.

Theorem (Kuwada, 2010)

$$\Gamma^{\mathbb{H}^1}(\mathsf{P}_t f) \leq C_{\mathbb{H}^1}^2\, \mathsf{P}_t \Gamma^{\mathbb{H}^1}(f) \iff W_2(\mathsf{P}_t\mu, \mathsf{P}_t\nu) \leq C_{\mathbb{H}^1}\, W_2(\mu,\nu)$$

Entropy

On (X, d), put a (non-negative, σ-finite and Borel) measure $\mathfrak{m}$. Assume that

$$\exists A, B > 0 \quad : \quad \mathfrak{m}\left(\{x \in X : \mathsf{d}(x, x_0) < r\}\right) \le A e^{Br^2}. \qquad \text{(exp.ball)}$$

The (Boltzmann) entropy $\mathsf{Ent}_{\mathfrak{m}} : \mathscr{P}_2(X) \to (-\infty, +\infty]$ is

$$\mathsf{Ent}_{\mathfrak{m}}(\mu) = \begin{cases} \displaystyle\int_X \varrho \log \varrho \, \mathrm{d}\mathfrak{m} & \text{if } \mu = \varrho \mathfrak{m} \in \mathscr{P}_2(X), \\ +\infty & \text{otherwise.} \end{cases}$$

Assumption (exp.ball) ensures that $\mathsf{Ent}(\mu) > -\infty$ for all $\mu \in \mathscr{P}_2(X)$.

If $(X, \mathsf{d}, \mathfrak{m}) = (\mathbb{M}, \mathsf{g})$ with $\mathrm{Ric} \ge K$, then

Bishop volume comparison Theorem $\Rightarrow$ (exp.ball).

If $(X, \mathsf{d}, \mathfrak{m}) = (\mathbb{H}^1, \mathsf{d}_{\mathsf{CC}}, \mathscr{L}^3)$, then

group structure and dilations $\Rightarrow$ (exp.ball),

because

$$\mathscr{L}^3(B_{\mathsf{CC}}(p, r)) = \mathscr{L}^3(B_{\mathsf{CC}}(0, r)) = \mathscr{L}^3(\delta_r(B_{\mathsf{CC}}(0, 1))) = r^4 \, \mathscr{L}^3(B_{\mathsf{CC}}(0, 1)).$$

Geodesic convexity of entropy and $\mathsf{CD}(K,\infty)$

Assume $(X,\mathsf{d},\mathfrak{m}) = (\mathbb{M},\mathsf{g})$. We have yet another equivalence with Ric $\geq K$.

Theorem (von Renesse - Sturm, 2005)

$$\mathrm{Ric} \geq K \iff \mathsf{Ent}_{\mathfrak{m}}(\mu_s) \leq (1-s)\mathsf{Ent}_{\mathfrak{m}}(\mu_0) + s\,\mathsf{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}s(1-s)\,W_2^2(\mu_0,\mu_1)$$

where $s \mapsto \mu_s$ is any (constant unit speed) W_2-geodesic joining μ_0 and μ_1.

Observation [Lott-Villani & Sturm]: the W_2-geodesic K-convexity of $\mathsf{Ent}_{\mathfrak{m}}$ does NOT need the smoothness of $(\mathbb{M},\mathsf{g})$, it ONLY needs d and $\mathfrak{m}$. Hence it makes sense in ANY metric-measure space.

Definition: $(X,\mathsf{d},\mathfrak{m})$ is $\mathsf{CD}(K,\infty)$ if $\mathsf{Ent}_{\mathfrak{m}}$ is W_2-geodesic K-convex.

Bad news: $(\mathbb{H}^1,\mathsf{d}_{CC},\mathscr{L}^3)$ does not satisfy the $\mathsf{CD}(K,\infty)$ property! [Juillet, 2009]

On $(\mathbb{H}^1,\mathsf{d}_{CC},\mathscr{L}^3)$ it actually holds [Balogh-Kristaly-Sipos, 2018]

$$\mathsf{Ent}_{\mathscr{L}^3}(\mu_s) \leq (1-s)\,\mathsf{Ent}_{\mathscr{L}^3}(\mu_0) + s\,\mathsf{Ent}_{\mathscr{L}^3}(\mu_1) + w(s)$$

where $w(s) = -2\log\left((1-s)^{(1-s)}s^s\right)$ for $s \in [0,1]$ (concave correction).

Heat flow in $(X, \mathsf{d}, \mathfrak{m})$

We now work in a metric-measure space $(X, \mathsf{d}, \mathfrak{m})$. The Cheeger energy is

$$\mathsf{Ch}(f) = \inf\left\{\liminf_n \int_X |\mathrm{D} f_n|^2 \, \mathrm{d}\mathfrak{m} : f_n \to f \text{ in } \mathrm{L}^2(X, \mathfrak{m}),\ f_n \in \mathrm{Lip}(X)\right\}.$$

Here $|\mathrm{D} f|(x) = \limsup_{y \to x} \frac{|f(y)-f(x)|}{\mathsf{d}(x,y)}$ denotes the slope of $f \in \mathrm{Lip}(X; \mathbb{R})$.

Properties: Cheeger energy is convex, l.s.c. and its domain $\mathrm{W}^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is dense.

The heat flow in $(X, \mathsf{d}, \mathfrak{m})$ is the (Hilbertian) gradient flow of Ch in $L^2(X, \mathfrak{m})$: for $f_0 \in L^2(X, \mathfrak{m})$, $\exists\, t \mapsto f_t = \mathsf{P}_t f_0 \in \mathrm{Lip}_{\mathrm{loc}}((0, +\infty); \mathrm{L}^2(X, \mathfrak{m}))$ such that

$$f_t \xrightarrow[t \to 0]{} f_0 \text{ in } \mathrm{L}^2(X, \mathfrak{m}) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} f_t \in -\partial^- \mathsf{Ch}(f_t) \text{ for a.e. } t > 0.$$

The Laplacian $-\Delta_{\mathsf{d},\mathfrak{m}} f \in \partial^- \mathsf{Ch}(f)$ the element of minimal $\mathrm{L}^2(X, \mathfrak{m})$-norm.

CAUTION: $\mathrm{W}^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $\|f\|_{\mathrm{W}^{1,2}} = \left(\|f\|_{\mathrm{L}^2}^2 + \mathsf{Ch}(f)\right)^{1/2}$ is Banach, but not Hilbert in general! For example, consider $(\mathbb{R}^n, \|\cdot\|_p, \mathscr{L}^n)$ for $p \neq 2$.

Theorem (Ambrosio - Gigli - Savaré, 2014)

Any $f \in \mathsf{W}^{1,2}(X, \mathsf{d}, \mathfrak{m})$ has a (unique) weak gradient $|\mathsf{D}f|_w \in \mathsf{L}^2(X, \mathfrak{m})$ such that

$$\mathsf{Ch}(f) = \frac{1}{2} \int_X |\mathsf{D}f|_w^2 \, \mathsf{d}\mathfrak{m}$$

Theorem (Ambrosio - Gigli - Savaré, 2014)

$|\mathsf{D}f|_w$ behaves like the 'modulus of the gradient':

- locality: $|\mathsf{D}f|_w = |\mathsf{D}g|_w$ $\mathfrak{m}$-a.e. on $\{f - g = c\}$;
- Leibniz rule: $|\mathsf{D}(fg)|_w \leq |f| \, |\mathsf{D}g|_w + |\mathsf{D}f|_w \, |g|$;
- chain rule: $|\mathsf{D}\varphi(f)|_w \leq |\varphi'(f)| \, |\mathsf{D}f|_w$;
- approximation: $\mathsf{Lip}_b(X) \cap \mathsf{L}^2(X, \mathfrak{m})$ is dense in energy in $\mathsf{W}^{1,2}(X, \mathsf{d}, \mathfrak{m})$.

Quadratic Cheeger energy

We say that Ch is quadratic if $\mathsf{Ch}(f+g)+\mathsf{Ch}(f-g)=2\mathsf{Ch}(f)+2\mathsf{Ch}(g)$.

Fact 1: Ch is quadratic $\Rightarrow W^{1,2}(X,\mathsf{d},\mathfrak{m})$ is Hilbert and P_t is linear.

Fact 2: Ch is quadratic $\Rightarrow \Gamma(f)=|\mathrm{D}f|_w^2$ is quadratic.

Theorem

$\Gamma(f,g)=|\mathrm{D}(f+g)|_w^2-|\mathrm{D}f|_w^2-|\mathrm{D}g|_w^2$ is 'the scalar product of gradients':

- Leibniz rule: $\Gamma(fg,h)=g\,\Gamma(f,h)+f\,\Gamma(g,h)$;
- chain rule: $\Gamma(\varphi(f),g)=\varphi'(f)\,\Gamma(f,g)$.

Theorem

If Ch is quadratic then the (Dirichlet) energy $\mathcal{E}(f)=2\mathsf{Ch}(f)$ satisfies

$$\mathcal{E}(f,g)=\int_X \Gamma(f,g)\,\mathrm{d}\mathfrak{m}=-\int_X g\,\Delta_{\mathsf{d},\mathfrak{m}}f\,\mathrm{d}\mathfrak{m}.$$

The Laplacian $\Delta_{\mathsf{d},\mathfrak{m}}$ satisfies the chain rule

$$\Delta_{\mathsf{d},\mathfrak{m}}(\varphi\circ f)=\varphi'(f)\,\Delta_{\mathsf{d},\mathfrak{m}}f+\varphi''(f)\,\Gamma(f).$$

Equivalence in $\mathsf{RCD}(K,\infty)$ spaces

Definition: $(X,\mathsf{d},\mathfrak{m})$ is $\mathsf{RCD}(K,\infty)$ if it is $\mathsf{CD}(K,\infty)$ and Ch is quadratic. [AGS]

Theorem (many people...)

Assume $(X,\mathsf{d},\mathfrak{m})$ has a quadratic Ch. TFAE:

$\mathsf{BE}(K,\infty)$: $\Gamma(\mathsf{P}_t f) \le e^{-2Kt}\,\mathsf{P}_t\Gamma(f)$

Kuwada: $W_2(\mathsf{P}_t\mu,\mathsf{P}_t\nu) \le e^{-Kt}\,W_2(\mu,\nu)$

$\mathsf{CD}(K,\infty)$: $\mathsf{Ent}_{\mathfrak{m}}(\mu_s) \le (1-s)\mathsf{Ent}_{\mathfrak{m}}(\mu_0) + s\,\mathsf{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}s(1-s)\,W_2^2(\mu_0,\mu_1)$

EVI_K: $\frac{\mathrm{d}}{\mathrm{d}t}\frac{W_2^2(\mathsf{P}_t\mu,\nu)}{2} + \frac{K}{2}W_2^2(\mathsf{P}_t\mu,\nu) + \mathsf{Ent}(\mathsf{P}_t\mu) \le \mathsf{Ent}(\nu)$

Remark: EVI_K stands for Evolution Variational Inequality and encodes the fact that the heat flow is the metric gradient flow of the entropy in the Wasserstein space.

Non-CD(K,∞) spaces: the Carnot groups

A Carnot group $\mathbb{G}$ is a connected, simply connected, stratified Lie group with

$$\mathsf{Lie}(\mathbb{G}) = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \quad V_i = [V_1, V_{i-1}], \quad [V_1, V_\kappa] = \{0\}.$$

By Campbell-Hausdorff formula, $\mathbb{G} \sim (\mathbb{R}^n, \cdot)$ using exponential coordinates.

We call $H\mathbb{G} = V_1$ the horizontal directions. If $V_1 = \mathrm{span}\{X_1, \dots, X_m\}$, then $\nabla_{\mathbb{G}} f = \sum_{j=1}^m (X_j f) X_j \in V_1$ and $\Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2$ (Kohn's sub-Laplacian).

The Carnot-Carathéodory distance of $x, y \in \mathbb{G}$ is

$$\mathsf{d}_{CC}(x,y) = \inf\left\{ \int_0^1 \|\dot\gamma_s\|_{\mathbb{G}}\, ds :\ \gamma_0 = x,\ \gamma_1 = y,\ \dot\gamma_t \in V_1 \right\}.$$

Then $(\mathbb{G}, \mathsf{d}_{cc}, \mathscr{L}^n)$ is Polish, geodesic and $\mathscr{L}^n(\mathsf{B}_{CC}(x,r)) = Cr^Q$, $Q \in \mathbb{N}$.

Example: for $\mathbb{H}^1$ it is $\kappa = 2$, $V_1 = \mathrm{span}\{X, Y\}$, $V_2 = \mathrm{span}\{Z\}$, $Q = 4$.

Theorem (Ambrosio - S., 2018)

The metric-measure space $(\mathbb{G}, \mathsf{d}_{CC}, \mathscr{L}^n)$ is not $\mathsf{CD}(K,\infty)$!

Another non-CD(K,∞) space: the $\mathbb{SU}(2)$ group

$\mathbb{SU}(2)$ = Lie group of 2×2 complex unitary matrices with determinant 1.

Lie algebra $\mathfrak{su}(2)$ = 2×2 complex unitary skew-Hermitian matrices with trace 0.

A basis of $\mathfrak{su}(2)$ is given by the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

satisfying the relations

$$[X,Y] = 2Z, \quad [Y,Z] = 2X, \quad [Z,X] = 2Y.$$

Similarly as before, we define d_{CC} and $\nabla_{\mathbb{SU}(2)} f = (Xf)X + (Yf)Y$.

<u>Fact</u>: $(\mathbb{SU}(2), \mathsf{d}_{CC})$ is a Polish geodesic metric space.

Using the cylindric coordinates (for $r \in [0,\frac{\pi}{2})$, $\vartheta \in [0,2\pi]$ and $\zeta \in [-\pi,\pi]$)

$$(r,\vartheta,z) \mapsto \exp(r\cos\vartheta\, X + r\sin\vartheta\, Y)\,\exp(\zeta\, Z) = \begin{pmatrix} e^{i\zeta}\cos r & e^{i(\vartheta-\zeta)}\sin r \\ -e^{-i(\vartheta-\zeta)}\sin r & e^{-i\zeta}\cos r \end{pmatrix}$$

the Haar measure $\sigma \in \mathscr{P}(\mathbb{SU}(2))$ is $\mathrm{d}\sigma = \frac{1}{4\pi^2}\sin(2r)\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\zeta$.

Why Carnot groups and $\mathbb{SU}(2)$?

Theorem (Melcher, 2008)

There exists $C_{\mathbb{G}} \geq 1$ such that $\Gamma^{\mathbb{G}}(\mathsf{P}_t f) \leq C_{\mathbb{G}}^2 \, \mathsf{P}_t \Gamma^{\mathbb{G}}(f)$.

Remark: $C_{\mathbb{G}} = 1 \iff \mathbb{G}$ is commutative [Ambrosio-S., 2018].

Theorem (Baudoin - Bonnefont, 2008)

There exists $C_{\mathbb{SU}(2)} \geq \sqrt{2}$ such that $\Gamma^{\mathbb{SU}(2)}(\mathsf{P}_t f) \leq C_{\mathbb{SU}(2)}^2 e^{-4t} \, \mathsf{P}_t \Gamma^{\mathbb{SU}(2)}(f)$.

Question: BE $\iff$ Kuwada $\iff$ RCD $\iff$ EVI also for $\mathbb{G}$ and $\mathbb{SU}(2)$?

Fact: [Kuwada, 2009] gives the equivalence with the W_2-contraction property.

Admissible metric-measure groups

Assume $(X, \mathsf{d}, \mathfrak{m})$ has Ch quadratic.

Definition (Admissible group)

$(X, \mathsf{d}, \mathfrak{m})$ is an admissible group if:

- the metric space (X, d) is locally compact;
- the set X is a topological group, i.e. $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous;
- d is left-invariant, i.e. $\mathsf{d}(zx, zy) = \mathsf{d}(x, y)$ for all $x, y, z \in X$;
- $\mathfrak{m}$ is a left-invariant Haar measure, i.e. $\mathfrak{m}$ is a Radon measure such that $\mathfrak{m}(xE) = \mathfrak{m}(E)$ for all $x \in X$ and all Borel set $E \subset X$;
- X is unimodular, i.e. $\mathfrak{m}$ is also right-invariant.

Remark: Carnot groups and $\mathbb{SU}(2)$ are admissible groups.

Main result

Let $\mathsf{c}\colon [0,+\infty) \to (0,+\infty)$ be such that $\mathsf{c}, \mathsf{c}^{-1} \in \mathsf{L}^\infty([0,T])$ for all $T > 0$.

Idea: c is the curvature function and generalizes $t \mapsto e^{-Kt}$.

Examples: for Carnot groups $\mathsf{c}(t) = C_{\mathbb{G}}$ and for $\mathbb{SU}(2)$ $\mathsf{c}(t) = C_{\mathbb{SU}(2)} e^{-2t}$.

Define $\mathsf{R}(a,b) = \int_0^1 \mathsf{c}^{-2}((1-s)a + sb)\, \mathsf{d}s$ for $0 \le a \le b$.

Theorem (S., 2020)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an admissible group + some technical hypotheses. TFAE:

BE_w: $\Gamma(\mathsf{P}_t f) \le \mathsf{c}^2(t)\, \mathsf{P}_t \Gamma(f)$

Kuwada: $W_2(\mathsf{P}_t\mu, \mathsf{P}_t\nu) \le \mathsf{c}(t)\, W_2(\mu,\nu)$

CD_w: $$\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \le (1-s)\,\mathsf{Ent}_m(\mathsf{P}_t\mu_0) + s\,\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1) + \frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)} W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0, \mathsf{P}_t\mu_1)\right)$$
for $t \ge 0$ and $h > 0$, with $s \mapsto \mu_s$ a W_2-geodesic

EVI_w: $$W_2^2(\mathsf{P}_{t_1}\mu_1, \mathsf{P}_{t_0}\mu_0) - \frac{W_2^2(\mu_1,\mu_0)}{\mathsf{R}(t_0,t_1)} \le 2(t_1 - t_0)\Big(\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$$
for $0 \le t_0 \le t_1$

Comments

$$\mathsf{CD}_w: \ \mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \le (1-s)\,\mathrm{Ent}_m(\mathsf{P}_t\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1) + \frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)}\,W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1)\right)$$

for $t \ge 0$ and $h > 0$

$$\mathsf{EVI}_w: \ W_2^2(\mathsf{P}_{t_1}\mu_1,\mathsf{P}_{t_0}\mu_0) - \frac{W_2^2(\mu_1,\mu_0)}{\mathsf{R}(t_0,t_1)} \le 2(t_1-t_0)\Big(\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$$

for $0 \le t_0 \le t_1$

1. The equivalence $\mathsf{BE}_w \iff$ Kuwada is known, see [Kuwada, 2009] and [Ambrosio - Gigli - Savaré, 2015], but we (re)do the proof because of some technical issues.
2. If $t = 0$ in CD_w then $\mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_h\mu_s) \le (1-s)\,\mathrm{Ent}_m(\mu_0) + s\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) + \frac{A(h)}{2}\,s(1-s)\,W_2^2(\mu_0,\mu_1)$ with $A(h) = \frac{\mathsf{R}(0,h)^{-1}-1}{h}$ for $h > 0$.
3. $\mathsf{CD}_w \Rightarrow$ Kuwada is easy: multiply by $h > 0$ and then send $h \to 0^+$.
4. $\mathsf{EVI}_w \Rightarrow \mathsf{CD}_w$ follows from a general argument, see [Daneri - Savaré, 2008].
5. We only need to prove $\mathsf{BE}_w \Rightarrow \mathsf{EVI}_w$. The proof is an adaptation of [Ambrosio - Gigli - Savaré, 2015] and [Erbar - Kuwada - Sturm, 2015].

Other comments and future developments

CD_w: $\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t+h}\mu_s) \leq (1-s)\,\mathsf{Ent}_m(\mathsf{P}_t\mu_0) + s\,\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_t\mu_1) + \frac{s(1-s)}{2h}\left(\frac{1}{\mathsf{R}(t,t+h)}\,W_2^2(\mu_0,\mu_1) - W_2^2(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1)\right)$
for $t \geq 0$ and $h > 0$

EVI_w: $W_2^2(\mathsf{P}_{t_1}\mu_1,\mathsf{P}_{t_0}\mu_0) - \frac{W_2^2(\mu_1,\mu_0)}{\mathsf{R}(t_0,t_1)} \leq 2(t_1-t_0)\Big(\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_0}\mu_0) - \mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{t_1}\mu_1)\Big)$
for $0 \leq t_0 \leq t_1$

1. We need the group structure of X to exploit the de-singularization property of the convolution: $\varrho \star \mu \ll \mathfrak{m}$. Can we avoid this assumption? Example: metric graphs.

Note: $\mathsf{BE}_w \Rightarrow \mathsf{P}_t\mu \ll \mathfrak{m}$, but the W_2-metric velocity of $s \mapsto \mu_s^t = \mathsf{P}_t\mu_s$ cannot be related to the one of $s \mapsto \mu_s$ if $\mathsf{c}(0+) > 1$ (example: Carnot groups and $\mathbb{SU}(2)$!).

2. Consider a sub-Riemannian manifold $\mathbb{M}$ (possibly, without a group structure). Is there a BE_w inequality also encoding information about the dimension of $\mathbb{M}$?

3. $\mathsf{RCD}(K,\infty)$ and EVI_K imply several nice properties about $(X,\mathsf{d},\mathfrak{m})$ (MCP, gradient flows, m-GH stability,...). What can we deduce from RCD_w and EVI_w?

Let $s \in [0,1]$ and assume $s \mapsto \mu_s = f_s \mathfrak{m}$ is joining $\mu_0, \mu_1 \in \mathscr{P}_2(X)$.

Define a new curve $s \mapsto \tilde{\mu}_s = \tilde{f}_s \mathfrak{m}$ as

$$\tilde{\mu}_s = \mathsf{P}_{\eta(s)} \mu_{\vartheta(s)}, \text{ so that } \tilde{f}_s = \mathsf{P}_{\eta(s)} f_{\vartheta(s)},$$

where $\eta \in \mathsf{C}^2([0,1];[0,+\infty))$ and $\vartheta \in \mathsf{C}^1([0,1];[0,1])$ with $\vartheta(0) = 0$ and $\vartheta(1) = 1$.

At least formally, we can compute

$$\frac{\mathsf{d}}{\mathsf{d}s} \tilde{f}_s = \dot{\eta}(s) \, \Delta \mathsf{P}_{\eta(s)} f_{\vartheta(s)} + \dot{\vartheta}(s) \, \mathsf{P}_{\eta(s)} \dot{f}_{\vartheta(s)}$$

for $s \in (0,1)$.

Proof of $\mathsf{BE}_w \Rightarrow \mathsf{EVI}_w$ [2/6]

On the one hand, integrating by parts, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Ent}_{\mathfrak{m}}(\tilde{\mu}_s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_X \tilde{f}_s \log \tilde{f}_s \,\mathrm{d}\mathfrak{m} \\ &= \int_X (1+\log \tilde{f}_s)\,\frac{\mathrm{d}}{\mathrm{d}s}\,\tilde{f}_s\,\mathrm{d}\mathfrak{m} \\ &= -\dot{\eta}(s)\int_X p'(\tilde{f}_s)\,\Gamma(\tilde{f}_s)\,\mathrm{d}\mathfrak{m} + \dot{\vartheta}(s)\int_X p(\tilde{f}_s)\,\mathsf{P}_{\eta(s)}\dot{f}_{\vartheta(s)}\,\mathrm{d}\mathfrak{m}\end{aligned}$$

for $s \in (0,1)$, where $p(r) = 1 + \log r$ for all $r > 0$.

Since $p'(r) = r(p'(r))^2$, by the chain rule $\Gamma(\varphi(f)) = (\varphi'(f))^2\,\Gamma(f)$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Ent}_{\mathfrak{m}}(\tilde{\mu}_s) = -\dot{\eta}(s)\int_X \Gamma(g_s)\,\mathrm{d}\tilde{\mu}_s + \dot{\vartheta}(s)\int_X \dot{f}_{\vartheta(s)}\,\mathsf{P}_{\eta(s)} g_s\,\mathrm{d}\mathfrak{m}$$

for $s \in (0,1)$, where $g_s = p(\tilde{f}_s)$ for brevity.

On the other hand, by Kantorovich duality, we have

$$\frac{1}{2} W_2^2(\mu,\nu) = \sup\left\{ \int_X Q_1\varphi \, \mathrm{d}\mu - \int_X \varphi \, \mathrm{d}\nu : \varphi \in \mathrm{Lip}(X) \text{ with bounded support} \right\},$$

where

$$Q_s\varphi(x) = \inf_{y\in X} \varphi(y) + \frac{\mathsf{d}^2(y,x)}{2}, \quad \text{for } x \in X \text{ and } s > 0,$$

is the Hopf-Lax infimum-convolution semigroup.

Recalling that $\varphi_s = Q_s\varphi$ solves the Hamilton-Jacobi equation $\partial_s\varphi_s + \frac{1}{2}\,|\mathrm{D}\varphi_s|^2 = 0$, again integrating by parts, we can compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s\,\tilde{f}_s\,\mathrm{d}\mathfrak{m} &= \int_X \partial_s\varphi_s\,\mathrm{d}\tilde{\mu}_s + \int_X \varphi_s\,\frac{\mathrm{d}}{\mathrm{d}s}\tilde{f}_s\,\mathrm{d}\mathfrak{m}\\ &= -\frac{1}{2}\int_X \Gamma(\varphi_s)\,\mathrm{d}\tilde{\mu}_s - \dot{\eta}(s)\int_X \Gamma(\varphi_s,\tilde{f}_s)\,\mathrm{d}\mathfrak{m}\\ &\quad + \dot{\vartheta}(s)\int_X \dot{f}_{\vartheta(s)}\,\mathsf{P}_{\eta(s)}\varphi_s\,\mathrm{d}\mathfrak{m}\end{aligned}$$

for $s \in (0,1)$.

Proof of $\mathsf{BE}_w \Rightarrow \mathsf{EVI}_w$ [4/6]

Combinining the above inequalities, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s\,\tilde{f}_s\,\mathrm{d}\mathfrak{m} + \dot{\eta}(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mathsf{Ent}_{\mathfrak{m}}(\tilde{\mu}_s) \leq -\frac{1}{2}\int_X \big(\Gamma(\varphi_s) + \dot{\eta}(s)^2\,\Gamma(g_s)\big)\,\mathrm{d}\tilde{\mu}_s$$
$$- \dot{\eta}(s)\int_X \Gamma(\varphi_s, \tilde{f}_s)\,\mathrm{d}\mathfrak{m} + \dot{\vartheta}(s)\int_X \dot{f}_{\vartheta(s)}\,\mathsf{P}_{\eta(s)}(\varphi_s + \dot{\eta}(s)\,g_s)\,\mathrm{d}\mathfrak{m}$$

for $s \in (0,1)$, forgetting the term $-\frac{\dot{\eta}(s)^2}{2}\int_X \Gamma(g_s)\,\mathrm{d}\tilde{\mu}_s \leq 0$.

Now $\Gamma(\varphi_s + \dot{\eta}(s)\,g_s) = \Gamma(\varphi_s) + 2\,\dot{\eta}(s)\,\Gamma(\varphi_s, g_s) + \dot{\eta}(s)^2\,\Gamma(g_s)$ and, by the chain rule, $\Gamma(\varphi_s, g_s) = \Gamma(\varphi_s, p(\tilde{f}_s)) = p'(\tilde{f}_s)\,\Gamma(\varphi_s, \tilde{f}_s)$. Since $r\,p'(r) = 1$, we have

$$\int_X \Gamma(\varphi_s, g_s)\,\mathrm{d}\tilde{\mu}_s = \int_X \tilde{f}_s\,p'(\tilde{f}_s)\,\Gamma(\varphi_s, \tilde{f}_s)\,\mathrm{d}\mathfrak{m} = \int_X \Gamma(\varphi_s, \tilde{f}_s)\,\mathrm{d}\mathfrak{m},$$

and thus the above inequality simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s\,\tilde{f}_s\,\mathrm{d}\mathfrak{m} + \dot{\eta}(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mathsf{Ent}_{\mathfrak{m}}(\tilde{\mu}_s) \leq -\frac{1}{2}\int_X \Gamma(\varphi_s + \dot{\eta}(s)\,g_s)\,\mathrm{d}\tilde{\mu}_s$$
$$+ \dot{\vartheta}(s)\int_X \dot{f}_{\vartheta(s)}\,\mathsf{P}_{\eta(s)}(\varphi_s + \dot{\eta}(s)\,g_s)\,\mathrm{d}\mathfrak{m}$$

for $s \in (0,1)$.

Proof of $\mathsf{BE}_w \Rightarrow \mathsf{EVI}_w$ [5/6]

At this point, the crucial information we need to know about the chosen curve $s \mapsto \mu_s = f_s \mathfrak{m}$ is that

$$\int_X \dot f_s \,\psi \,\mathsf{d}\mathfrak{m} \le |\dot\mu_s| \left(\int_X \Gamma(\psi)\,\mathsf{d}\mu_s\right)^{\frac{1}{2}}$$

for all sufficiently 'nice' functions ψ, where $|\dot\mu_s| = \lim_{h\to 0} \frac{W_2(\mu_{s+h},\mu_s)}{h}$ is the metric velocity of the curve $s \mapsto \mu_s$ with respect to the Wasserstein distance.

With this property at disposal, we may choose $\psi = \mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\,g_s)$ and estimate

$$\begin{aligned}
\dot\vartheta(s)\int_X \dot f_{\vartheta(s)}\,\mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\,g_s)\,\mathsf{d}\mathfrak{m} &= \int_X \left(\frac{\mathsf{d}}{\mathsf{d}s} f_{\vartheta(s)}\right) \mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\,g_s)\,\mathsf{d}\mathfrak{m} \\
&\le |\dot\vartheta(s)|\,|\dot\mu_{\vartheta(s)}| \left(\int_X \Gamma(\mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\,g_s))\,\mathsf{d}\mu_s\right)^{\frac{1}{2}} \\
&\le \frac{\mathsf{c}^2(\eta(s))}{2}\,\dot\vartheta(s)^2\,|\dot\mu_{\vartheta(s)}|^2 + \frac{\mathsf{c}^{-2}(\eta(s))}{2}\int_X \Gamma(\mathsf{P}_{\eta(s)}(\varphi_s + \dot\eta(s)\,g_s))\,\mathsf{d}\mu_s \\
&\le \frac{\mathsf{c}^2(\eta(s))}{2}\,\dot\vartheta(s)^2\,|\dot\mu_{\vartheta(s)}|^2 + \frac{1}{2}\int_X \Gamma(\varphi_s + \dot\eta(s)\,g_s)\,\mathsf{d}\tilde\mu_s.
\end{aligned}$$

Proof of $\mathsf{BE}_w \Rightarrow \mathsf{EVI}_w$ [6/6]

By combining the above inequalities, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s\,\tilde{f}_s\,\mathrm{d}\mathfrak{m} + \dot\eta(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mathsf{Ent}_{\mathfrak{m}}(\tilde\mu_s) \le \frac{\mathsf{c}^2(\eta(s))}{2}\,\dot\vartheta(s)^2\,|\dot\mu_{\vartheta(s)}|^2$$

for $s \in (0,1)$.

If we choose $\dot\vartheta(s) = \mathsf{c}^{-2}(\eta(s))$, then we can integrate in $s \in (0,1)$ so that, by Kantorovich duality, we finally get

$$\begin{aligned}\frac{1}{2}\,W_2^2(\mathsf{P}_{\eta(1)}\mu_1, \mathsf{P}_{\eta(0)}\mu_0) - \frac{1}{2\,\mathsf{R}(\eta)}\,W_2^2(\mu_1,\mu_0) + \dot\eta(1)\,\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{\eta(1)}\mu_1)\\ \le \dot\eta(0)\,\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{\eta(0)}\mu_0) + \int_0^1 \ddot\eta(s)\,\mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{\eta(s)}\mu_{\vartheta(s)})\,\mathrm{d}s,\end{aligned}$$

where $\mathsf{R}(\eta) = \int_0^1 \mathsf{c}^{-2}(\eta(s))\,\mathrm{d}s$.

Since we have no information about $s \mapsto \mathsf{Ent}_{\mathfrak{m}}(\mathsf{P}_{\eta(s)}\mu_{\vartheta(s)})$, we choose $\eta(s) = (1-s)t_0 + st_1$ for $s \in [0,1]$, where $0 \le t_0 \le t_1$ are fixed, and we get EVI_w.

Thank you for your attention!