A Distributional Approach to Fractional Sobolev Spaces and Fractional Variation

Candidato: Giorgio Stefani

Relatore: Prof. Luigi Ambrosio

Scuola Normale Superiore

Pisa, 13 February 2020

G. E. Comi and G. Stefani, "A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up", J. Funct. Anal. 277 (2019), no. 10, 3373-3435.

G. E. Comi and G. Stefani, "A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics I" (2019), submitted, available at arXiv:1910.13419.

E. Bruè, M. Calzi, G. E. Comi and G. Stefani, "A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics II" (2019), in preparation.

No paradoxes without utility

Around 1675 Newton and Leibniz discovered Calculus and nowadays derivative is a basic tool of any mathematician.

Somewhat surprisingly, the first appearance of the concept of a fractional derivative is found in a letter written to De l'Hôpital by Leibniz in 1695!

What is the "half derivative" of x? It's $\dfrac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = c\sqrt{x}$ (with $c = \frac{2}{\sqrt{\pi}}$ by Lacroix, 1819).

Leibniz's answer to De L'Hôpital, 30 September 1695:

"Il y a de l'apparence qu'on tirera un jour des consequences bien utiles de ces paradoxes, car il n'y a gueres de paradoxes sans utilité."

"This is an apparent paradox from which, one day, useful consequences will be drawn, since there are no paradoxes without utility."

Three famous examples

Let us recall three famous fractional derivatives:

Leibniz-Lacroix (1819): $$\frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Riemann-Liouville (1832-1847): $${}^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha}\, d\tau$$

Caputo (1967): $${}^{C}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau.$$

Some observations:

- they are defined just for functions of one variable;
- only Caputo's derivative kills constants;
- Caputo's derivative requires f to be differentiable!

Question: What about fractional gradient? Can we just take $\left(D^{\alpha,1}, \dots, D^{\alpha,n}\right)$?

Be careful: the "coordinate approach" gives an operator not invariant by rotations!

Šilhavý's approach: invariance properties

Recently, Šilhavý proposed that a "good" fractional operator should satisfy:

- invariance with respect to translations and rotations;
- α-homogeneity for some $\alpha \in (0,1)$;
- mild continuity on suitable test space, e.g. C_c^∞ or Schwartz's space $\mathscr{S}$.

Idea behind: fractional operators should have a physical meaning!

For $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we consider

$$\nabla^\alpha f(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(y-x)}{|y-x|^{n+\alpha+1}}\,dy, \quad x \in \mathbb{R}^n,$$

and

$$\operatorname{div}^\alpha \varphi(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy, \quad x \in \mathbb{R}^n.$$

Theorem (Šilhavý, 2020)

∇^α and $\operatorname{div}^\alpha$ are determined (up to mult. const.) by the three requirements above.

A little bit of literature

The operator $\nabla^\alpha \equiv \nabla I_{1-\alpha}$ has in fact a long story:

- Horváth, 1959 (earliest reference up to knowledge);
- implicitly mentioned in Nikol'ski-Sobolev, 1961;
- non-local continuum mechanics by Edelen-Green-Laws, 1971;
- non-local porous medium equation Caffarelli-Soria-Vazquez, 2011-13, and Biler-Imbert-Karch, 2015;
- fractional PDE theory and "geometric" inequalities by Shieh-Spector, Ponce-Spector, Schikorra-Spector-Van Schaftingen, all after 2015;
- distributional approach by Šilhavý, 2020.

Duality, fractional Laplacian and Riesz transform

The operators ∇^α and div^α are dual, in the sense that

$$\int_{\mathbb{R}^n} f \, \mathrm{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

The operators ∇^α and div^β satisfy $-\mathrm{div}^\beta \nabla^\alpha = (-\Delta)^{\frac{\alpha+\beta}{2}}$.

If we let

$$I_\alpha u(x) := \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}} \, dy$$

be the fractional Riesz potential of $u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^m)$, then

$$\nabla^\alpha f = I_{1-\alpha} \nabla f, \qquad \mathrm{div}^\alpha \varphi = I_{1-\alpha} \, \mathrm{div} \varphi.$$

Integrability: $\nabla^\alpha f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\mathrm{div}^\alpha \varphi \in L^1(\mathbb{R}^n; \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

Relax: ∇^α and div^α are well posed also for Lip_c-regular test functions.

Leibniz's rules for ∇^α and $\operatorname{div}^\alpha$

For any $f, g \in C_c^\infty(\mathbb{R}^n)$, we have it holds

$$\nabla^\alpha(fg) = f\nabla^\alpha g + g\nabla^\alpha f + \nabla^\alpha_{\mathrm{NL}}(f,g),$$

where

$$\nabla^\alpha_{\mathrm{NL}}(f,g)(x) := \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(g(y)-g(x))(y-x)}{|y-x|^{n+\alpha+1}}\,dy, \quad x\in\mathbb{R}^n.$$

For any $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we also have it holds

$$\operatorname{div}^\alpha(f\varphi) = f\operatorname{div}^\alpha\varphi + \varphi\cdot\nabla^\alpha f + \operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi),$$

where

$$\operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi)(x) := \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy, \quad x\in\mathbb{R}^n.$$

Although ∇^α and $\operatorname{div}^\alpha$ have a strong non-local behaviour, they commute with convolutions (by linearity). In addition, Leibniz's rules allow for approximation by cut-off functions (careful control on non-local terms).

A fractional version of the Fundamental Theorem of Calculus

Let $\alpha \in (0,1)$. If $f \in C_c^\infty(\mathbb{R}^n)$, then

$$f(y) - f(x) = \mu_{n,-\alpha} \int_{\mathbb{R}^n} \left(\frac{z-x}{|z-x|^{n+1-\alpha}} - \frac{z-y}{|z-y|^{n+1-\alpha}} \right) \cdot \nabla^\alpha f(z)\, dz$$

for any $x, y \in \mathbb{R}^n$.

Some good news:

- we get L^1-control on translations;
- we get L^1-control on smoothed-by-convolution functions;
- we get compactness for sequences with uniformly bounded RHS.

Some bad news:

- left-hand integral is on the whole space (non-locality!);
- we cannot get local Poincaré inequality;
- we cannot get relative fractional isoperimetric inequality.

We define

$$BV^\alpha(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty \right\},$$

where

$$|D^\alpha f|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1 \right\}.$$

In perfect analogy with the classical BV framework:

- $BV^\alpha(\mathbb{R}^n)$ is a Banach space and its norm is l.s.c. w.r.t. L^1-convergence;
- $C^\infty(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ are dense subspaces of $BV^\alpha(\mathbb{R}^n)$;
- given $f \in L^1(\mathbb{R}^n)$, $f \in BV^\alpha(\mathbb{R}^n) \iff \exists D^\alpha f \in \mathcal{M}(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f$$

  for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$;
- unif. bounded seq. in $BV^\alpha(\mathbb{R}^n)$ admit limit points in $L^1(\mathbb{R}^n)$ w.r.t. L^1_{loc}-conv.;
- for $n \ge 2$ we have $BV^\alpha(\mathbb{R}^n) \subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ (Gagliardo-Nirenberg-Sobolev).

Fractional distributional Sobolev spaces and Bessel potential spaces

For $p \in [1, +\infty]$, we define the distributional fractional Sobolev space

$$S^{\alpha,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : \exists\, \nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n)\}.$$

Here $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ is the weak fractional gradient of $f \in L^p(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} f \, \mathrm{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n).$$

We naturally have $S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ with

$$f \in S^{\alpha,1}(\mathbb{R}^n) \iff |D^\alpha f| \ll \mathscr{L}^n, \; D^\alpha f = \nabla^\alpha f \, \mathscr{L}^n.$$

We are also able to prove that $BV^\alpha(\mathbb{R}^n) \setminus S^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$.

For $p \in (1, +\infty)$, we prove that $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$, where

$$L^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : (I - \Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n)\}$$

is the Bessel potential space.

Fractional Sobolev spaces and fractional operators

For $p \in [1, +\infty)$ and $\alpha \in (0, 1)$, we let

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\alpha}} \, dx \, dy < +\infty \right\}.$$

The fractional perimeter in an open set $\Omega \subset \mathbb{R}^n$ of a measurable set $E \subset \mathbb{R}^n$ is

$$P_\alpha(E; \Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+\alpha}} \, dx \, dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+\alpha}} \, dx \, dy.$$

If $\Omega = \mathbb{R}^n$, then $P_\alpha(E; \mathbb{R}^n) = P_\alpha(E) = [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)}$.

Notice that we have the extension $\nabla^\alpha \colon W^{\alpha,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n; \mathbb{R}^n)$, since

$$\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \leq \mu_{n,\alpha} [f]_{W^{\alpha,1}(\mathbb{R}^n)} \text{ for all } f \in W^{\alpha,1}(\mathbb{R}^n).$$

We thus have $W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n)$ with $f \in W^{\alpha,1}(\mathbb{R}^n) \Rightarrow D^\alpha f = \nabla^\alpha f \mathscr{L}^n$.

Since $W^{\alpha,1}(\mathbb{R}^n)$ is closed w.r.t. pointwise convergence, $S^{\alpha,1}(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$.

Remarkably, if $0 < \beta < \alpha < 1$ then $BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$.

Sets of finite fractional Caccioppoli α-perimeter

In perfect analogy with the standard BV setting, we give the following definition.

Let $\alpha \in (0,1)$ and $E \subset \mathbb{R}^n$ be a measurable set. For any open set $\Omega \subset \mathbb{R}^n$, we let

$$|D^\alpha \chi_E|(\Omega) = \sup\left\{ \int_E \mathrm{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \; \|\varphi\|_{L^\infty(\Omega; \mathbb{R}^n)} \le 1 \right\}$$

be the fractional Caccioppoli α-perimeter of E in Ω. If $|D^\alpha \chi_E|(\Omega) < +\infty$, then E has finite fractional Caccioppoli α-perimeter in Ω.

Note that $E \subset \mathbb{R}^n$ has finite fractional Caccioppoli α-perimeter in Ω if and only if $D^\alpha \chi_E \in \mathcal{M}(\Omega; \mathbb{R}^n)$ and

$$\int_E \mathrm{div}^\alpha \varphi \, dx = -\int_\Omega \varphi \cdot dD^\alpha \chi_E$$

for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$.

Question: can we define a fractional version of De Giorgi's reduce boundary?

Fractional reduced boundary

It is now natural to give the following definition.

Let $E \subset \mathbb{R}^n$ be a set with finite fractional Caccioppoli α-perimeter in Ω. A point $x \in \Omega$ belongs to the fractional reduced boundary of E (inside Ω) if

$$x \in \operatorname{supp}(D^\alpha \chi_E) \qquad \text{and} \qquad \exists \lim_{r\to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} \in \mathbb{S}^{n-1}.$$

We thus let $\mathscr{F}^\alpha E$ be the fractional reduced boundary of E and define

$$\nu_E^\alpha \colon \Omega \cap \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}, \qquad \nu_E^\alpha(x) := \lim_{r\to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))}, \quad x \in \Omega \cap \mathscr{F}^\alpha E,$$

the inner unit fractional normal to E (inside Ω).

We thus have the following Gauss-Green formula

$$\int_E \operatorname{div}^\alpha \varphi \, dx = -\int_{\Omega \cap \mathscr{F}^\alpha E} \varphi \cdot \nu_E^\alpha \, d|D^\alpha \chi_E|.$$

for all $\varphi \in \operatorname{Lip}_c(\Omega; \mathbb{R}^n)$.

Sets of finite fractional perimeter

If $E \subset \mathbb{R}^n$ satisfies $P_\alpha(E;\Omega) < +\infty$, then

$$|D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E;\Omega)$$

and

$$D^\alpha \chi_E = \nu_E^\alpha \, |D^\alpha \chi_E| = \nabla^\alpha \chi_E \, \mathscr{L}^n.$$

Moreover, if $\chi_E \in BV(\mathbb{R}^n)$, then

$$\nabla^\alpha \chi_E(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{\nu_E(y)}{|y-x|^{n+\alpha-1}} \, d|D\chi_E|(y)$$

for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Be careful! We have

$$P_\alpha(E;\Omega) < +\infty \Rightarrow \mathscr{L}^n(\Omega \cap \mathscr{F}^\alpha E) > 0$$

including even the case $\chi_E \in BV(\mathbb{R}^n)$. In other words, the non-local operator ∇^α produces a diffuse fractional boundary in the $W^{\alpha,1}$ regime ($\subset S^{\alpha,1}$).

Example: $E = (a,b) \subset \mathbb{R} \Rightarrow \mathscr{F}^\alpha E = \mathbb{R} \setminus \left\{\frac{a+b}{2}\right\}$!

Two examples: balls and halfspaces

Example 1. For $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$, we have

$$\nabla^\alpha \chi_{B_1}(x) = -\frac{\mu_{n,\alpha}}{n+\alpha-1} g_{n,\alpha}(|x|) \frac{x}{|x|},$$

where

$$g_{n,\alpha}(t) := \int_{\partial B_1} \frac{y_1}{|t\mathsf{e}_1 - y|^{n+\alpha-1}} \, d\mathscr{H}^{n-1}(y) > 0, \quad \text{for any } t \geq 0,$$

which means $\nu^\alpha_{B_1}(x) = -x/|x|$ for any $x \neq 0$ and $\mathscr{F}^\alpha B_1 = \mathbb{R}^n \setminus \{0\}$.

Example 2. For the halfspace $H^+_\nu = \{y \cdot \nu \geq 0\}$, if $x \cdot \nu \neq 0$ then

$$\nabla^\alpha \chi_{H^+_\nu}(x) = \frac{2^{\alpha-1}\Gamma\left(\frac{\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1-\alpha}{2}\right)} \frac{1}{|x \cdot \nu|^\alpha} \nu.$$

In particular, $\mathscr{F}^\alpha H^+_\nu = \mathbb{R}^n$ and $\nu^\alpha_{H^+_\nu} \equiv \nu$.

Density estimates

Thanks to the invariance properties, we get
$$D^\alpha \chi_{\frac{E-x}{r}} = \frac{1}{r^{n-\alpha}} (I_{x,r})_\# D^\alpha \chi_E,$$
where $I_{x,r}(y) = (y-x)/r$ for $x, y \in \mathbb{R}^n$ and $r > 0$.

Theorem (Comi-S., 2019)

There exist $A_{n,\alpha}, B_{n,\alpha} > 0$ as follows. If $E \subset \mathbb{R}^n$ has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, then for any $x \in \mathscr{F}^\alpha E$ there exists $r_x > 0$ such that
$$|D^\alpha \chi_E|(B_r(x)) \le A_{n,\alpha} r^{n-\alpha}, \quad |D^\alpha \chi_{E \cap B_r(x)}|(\mathbb{R}^n) \le B_{n,\alpha} r^{n-\alpha}$$
for all $r \in (0, r_x)$.

By a standard covering argument, we thus get that
$$|D^\alpha \chi_E| \le C_{n,\alpha} \mathscr{H}^{n-\alpha} \llcorner \mathscr{F}^\alpha E$$
for some $C_{n,\alpha} > 0$ and, consequently,
$$\dim_{\mathscr{H}}(\mathscr{F}^\alpha E) \ge n - \alpha.$$

Existence of blow-ups and coarea inequality

Let Tan(E, x) be the set of all tangent sets of E at x, i.e. the set of all limit points in $L^1_{\text{loc}}(\mathbb{R}^n)$-topology of the family

$$\left\{\frac{E-x}{r} : r > 0\right\} \quad \text{as } r \to 0.$$

Theorem (Comi-S., 2019)

If E has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, then Tan$(E, x) \neq \varnothing$ for all $x \in \mathscr{F}^\alpha E$. Moreover, if $F \in$ Tan(E, x), then F has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ and $\nu_F^\alpha(y) = \nu_E^\alpha(x)$ for $|D^\alpha \chi_F|$-a.e. $y \in \mathscr{F}^\alpha F$.

What is missing: density estimates from below and we need coarea fromula.

Theorem (Comi-S., 2019)

If $f \in BV^\alpha(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|(\mathbb{R}^n)\, dt < +\infty$, then

$$D^\alpha f = \int_{\mathbb{R}} D^\alpha \chi_{\{f>t\}}\, dt, \qquad |D^\alpha f| \leq \int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|\, dt.$$

Bad news: there exist $f \in BV^\alpha(\mathbb{R}^n)$ such that $\int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|(\mathbb{R}^n)\, dt = +\infty$!

Asymptotics as $\alpha \to 1^-$

Theorem (Maz'ya-Shaposhnikova & Davila, 2002)

Let $p \in [1, +\infty)$. If $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} (1-\alpha)[f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = c_{n,p} \|\nabla f\|^p_{L^p(\mathbb{R}^n;\, \mathbb{R}^n)}.$$

If $f \in BV(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} (1-\alpha)[f]_{W^{\alpha,1}(\mathbb{R}^n)} = c_{n,1} |Df|(\mathbb{R}^n).$$

Now it is important to observe that

$$\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \sim \frac{1-\alpha}{\omega_n} \quad \text{as } \alpha \to 1^-.$$

Theorem (Comi-S., 2019)

Let $p \in [1, +\infty)$. If $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} \|\nabla^\alpha f - \nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0.$$

If $f \in BV(\mathbb{R}^n)$, then $D^\alpha f \rightharpoonup Df$ and $|D^\alpha f| \rightharpoonup |Df|$ as $\alpha \to 1^-$ and moreover

$$\lim_{\alpha \to 1^-} |D^\alpha f|(\mathbb{R}^n) = |Df|(\mathbb{R}^n).$$

Γ-convergence as $\alpha \to 1^-$

Theorem (Ambrosio-De Philippis-Martinazzi, 2011)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$\Gamma(L^1_{\text{loc}})\text{-}\lim_{\alpha \to 1^-} (1-\alpha) P_\alpha(E;\Omega) = 2\omega_{n-1} P(E;\Omega)$$

for every measurable set $E \subset \mathbb{R}^n$.

Theorem (Comi-S., 2019)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$\Gamma(L^1_{\text{loc}})\text{-}\lim_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) = P(E;\Omega)$$

for every measurable set $E \subset \mathbb{R}^n$.

Theorem (Comi-S., 2019)

Let $\Omega \subset \mathbb{R}^n$ be an open set such that Ω is bounded with Lipschitz boundary or $\Omega = \mathbb{R}^n$. Then

$$\Gamma(L^1)\text{-}\lim_{\alpha \to 1^-} |D^\alpha f|(\Omega) = |Df|(\Omega)$$

for every $f \in BV(\mathbb{R}^n)$.

The space $BV^0(\mathbb{R}^n)$ and the Hardy space

Imitating what we did before, we define

$$BV^0(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |D^0 f|(\mathbb{R}^n) < +\infty\},$$

where

$$|D^0 f|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^0\varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\}.$$

Here div^0 is the dual operator of $\nabla^0 = I_1\nabla = R$, the Riesz transform

$$Rf(x) = \mu_{n,0} \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(y - x)}{|y - x|^{n+1}} \, dy, \quad x \in \mathbb{R}^n,$$

(in the principal value sense).

We can prove that $BV^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, where

$$H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : Rf \in L^1(\mathbb{R}^n;\mathbb{R}^n)\}$$

is the (real) Hardy space, with $D^0 f = Rf \, \mathscr{L}^n$ as measures for $f \in BV^0(\mathbb{R}^n)$.

Asymptotics as $\alpha \to 0^+$

Theorem (Maz'ya-Shaposhnikova, 2002)

Let $p \in [1, +\infty)$. If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \alpha\, [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = c_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$

Now it is important to observe that $\mu_{n,\alpha} \to \mu_{n,0}$ as $\alpha \to 0^+$ (no rescaling!).

Theorem (Bruè-Calzi-Comi-S., in preparation)

Let $p \in (1, +\infty)$. If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \|\nabla^\alpha f - Rf\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0.$$

If $f \in H^1(\mathbb{R}^n) \cap \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then (actually, with H^1 norm)

$$\lim_{\alpha \to 0^+} \|\nabla^\alpha f - Rf\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} = 0.$$

If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \alpha \int_{\mathbb{R}^n} |\nabla^\alpha f(x)|\, dx = n\omega_n \mu_{n,0} \left| \int_{\mathbb{R}^n} f(x)\, dx \right|.$$

Fractional interpolation inequalities

We prove $\nabla^\alpha \to R$ strongly as $\alpha \to 0^+$ via fractional interpolation inequalities.

Theorem (Bruè-Calzi-Comi-S., in preparation)

Let $\alpha \in (0,1]$. There exists $c_{n,\alpha} > 0$ such that

$$|D^\beta f|(\mathbb{R}^n) \le c_{n,\alpha} \|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} |D^\alpha f|(\mathbb{R}^n)^{\frac{\beta}{\alpha}}$$

for all $\beta \in [0,\alpha]$ and all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$.

Theorem (Bruè-Calzi-Comi-S., in preparation)

Let $p \in (1,+\infty)$. There exists $c_{n,p} > 0$ such that

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,p} \|\nabla^\gamma f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$. There exists $c_n > 0$ such that

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)} \le c_n \|\nabla^\gamma f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in HS^{\alpha,1}(\mathbb{R}^n)$ (i.e., $f \in H^1$ and $\nabla^\alpha f \in H^1$).

Open problems and research directions [I/2]

About sets and perimeter. With the distributional approach to fractional variation, many research directions are interesting.

▷ We proved that $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ strictly only at the level of functions. Is there $\chi_E \in BV^\alpha(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n)$ for some measurable set $E \subset \mathbb{R}^n$?

▷ We know that blow-ups exist and have constant fractional normal. Can we characterise them more precisely? Are they unique (in some cases)?

▷ Minimal surfaces for P_α are widely studied. What about minimal surfaces for the fractional variation? Can we perform calibrations?

▷ Isoperimetric sets for P_α are balls (also in a quantitative sense). Are balls isoperimetric sets for the fractional variation?

Open problems and research directions [2/2]

About interpolation. Fractional interpolation inequalities may be derived from real/complex Interpolation Theory.

- ▷ What is the real interpolation space between $BV(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$? Note that

$$(H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\alpha,p} \subset (L^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\alpha,p} = B^\alpha_{1,p}(\mathbb{R}^n) \supset W^{\alpha,1}(\mathbb{R}^n)$$

and that $B^\alpha_{1,1}(\mathbb{R}^n) = W^{\alpha,1}(\mathbb{R}^n)$.

- ▷ What is the complex interp. space between $BV(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$? Note that

$$\begin{array}{ccccc} (H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\alpha,1} & \subset & (H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{[\alpha]} & \subset & (H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\alpha,\infty} \\ \cap & & & & \cap \\ W^{\alpha,1}(\mathbb{R}^n) & \subsetneq & BV^\alpha(\mathbb{R}^n) & \subsetneq & B^\alpha_{1,\infty}(\mathbb{R}^n) \end{array}$$

(with the second inclusion strict for all $n \geq 2$).

About the general theory. What is the "right" definition of BV^α on a general open set $\Omega \subset \mathbb{R}^n$? We would like to keep integration by parts, but what is the role of $\partial\Omega$?

Thank you for your attention!