A distributional approach to fractional Sobolev spaces and fractional variation: asymptotic analysis

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Fractional derivative: three famous examples

Around 1675 Newton and Leibniz discovered Calculus. Somewhat surprisingly, the first appearance of the concept of a fractional derivative is found in a letter written to De l'Hôpital by Leibniz in 1695!

Let us recall the three most famous fractional derivatives:

Leibniz-Lacroix (1819): $$\frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Riemann-Liouville (1832-1847): $${}^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha}\, d\tau$$

Caputo (1967): $${}^{C}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau.$$

Some observations:

- they are defined just for functions of one variable;
- only Caputo's derivative kills constants;
- Caputo's derivative requires f to be differentiable!

<u>Question</u>: What about fractional gradient? Can we just take $\left(D^{\alpha,1}, \dots, D^{\alpha,n}\right)$?

<u>Be careful</u>: the "coordinate approach" gives an operator not invariant by rotations!

Šilhavý's approach: invariance properties

Recently, Šilhavý proposed that a "good" fractional operator should satisfy:

- invariance with respect to translations and rotations;
- α-homogeneity for some $\alpha \in (0,1)$;
- mild continuity on suitable test space, e.g. C_c^∞ or Schwartz's space $\mathscr{S}$.

Idea behind: fractional operators should have a physical meaning!

For $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we consider

$$\nabla^\alpha f(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(y-x)}{|y-x|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n,$$

and

$$\operatorname{div}^\alpha \varphi(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n.$$

Theorem (Šilhavý, 2020)

∇^α and $\operatorname{div}^\alpha$ are determined (up to mult. const.) by the three requirements above.

A little bit of literature

The operator $\nabla^\alpha \equiv \nabla I_{1-\alpha}$ has in fact a long story:

- Horváth, 1959 (earliest reference up to knowledge);
- implicitly mentioned in Nikol'ski-Sobolev, 1961;
- non-local continuum mechanics by Edelen-Green-Laws, 1971;
- non-local porous medium equation Caffarelli-Soria-Vazquez, 2011-13, and Biler-Imbert-Karch, 2015;
- fractional PDE theory and "geometric" inequalities by Shieh-Spector, Ponce-Spector, Schikorra-Spector-Van Schaftingen, all after 2015;
- distributional approach by Šilhavý, 2020.

The operators ∇^α and div^α are dual, in the sense that

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

The operators ∇^α and div^β satisfy $-\mathrm{div}^\beta \nabla^\alpha = (-\Delta)^{\frac{\alpha+\beta}{2}}$.

If we let

$$I_\alpha u(x) := \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}} \, dy$$

be the fractional Riesz potential of $u \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m)$, then

$$\nabla^\alpha f = I_{1-\alpha} \nabla f, \qquad \mathrm{div}^\alpha \varphi = I_{1-\alpha}\, \mathrm{div} \varphi.$$

Integrability: $\nabla^\alpha f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\mathrm{div}^\alpha \varphi \in L^1(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

Relax: ∇^α and div^α are well posed also for Lip_c-regular test functions.

Fractional variation and the space $BV^\alpha(\mathbb{R}^n)$

We define

$$BV^\alpha(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty\},$$

where

$$|D^\alpha f|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f\, \mathrm{div}^\alpha \varphi\, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\}.$$

In perfect analogy with the classical BV framework:

- $BV^\alpha(\mathbb{R}^n)$ is a Banach space and its norm is l.s.c. w.r.t. L^1-convergence;
- $C^\infty(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ are dense subspaces of $BV^\alpha(\mathbb{R}^n)$;
- given $f \in L^1(\mathbb{R}^n)$, $f \in BV^\alpha(\mathbb{R}^n) \iff \exists D^\alpha f \in \mathcal{M}(\mathbb{R}^n;\mathbb{R}^n)$ such that

  $$\int_{\mathbb{R}^n} f\ \mathrm{div}^\alpha \varphi\, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f$$

  for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$;
- unif. bounded seq. in $BV^\alpha(\mathbb{R}^n)$ admit limit points in $L^1(\mathbb{R}^n)$ w.r.t. L^1_{loc}-conv.;
- for $n \ge 2$ we have $BV^\alpha(\mathbb{R}^n) \subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ (Gagliardo-Nirenberg-Sobolev).

Fractional distributional Sobolev spaces and Bessel potential spaces

For $p \in [1, +\infty]$, we define the distributional fractional Sobolev space

$$S^{\alpha,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : \exists\, \nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n)\}.$$

Here $\nabla^\alpha f \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ is the weak fractional gradient of $f \in L^p(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} f \, \text{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n).$$

We naturally have $S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ with

$$f \in S^{\alpha,1}(\mathbb{R}^n) \iff |D^\alpha f| \ll \mathscr{L}^n, \; D^\alpha f = \nabla^\alpha f \, \mathscr{L}^n.$$

We are also able to prove that $BV^\alpha(\mathbb{R}^n) \setminus S^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$.

For $p \in (1, +\infty)$, we prove that $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$, where

$$L^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : (I - \Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n)\}$$

is the Bessel potential space.

Fractional Sobolev spaces and fractional operators

For $p \in [1, +\infty)$ and $\alpha \in (0,1)$, we let

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+p\alpha}} \, dx \, dy < +\infty \right\}.$$

The fractional perimeter in an open set $\Omega \subset \mathbb{R}^n$ of a measurable set $E \subset \mathbb{R}^n$ is

$$P_\alpha(E;\Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}} \, dx \, dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}} \, dx \, dy.$$

If $\Omega = \mathbb{R}^n$, then $P_\alpha(E;\mathbb{R}^n) = P_\alpha(E) = [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)}$.

Notice that we have the extension $\nabla^\alpha \colon W^{\alpha,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n;\mathbb{R}^n)$, since

$$\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \mu_{n,\alpha} [f]_{W^{\alpha,1}(\mathbb{R}^n)} \text{ for all } f \in W^{\alpha,1}(\mathbb{R}^n).$$

We thus have $W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n)$ with $f \in W^{\alpha,1}(\mathbb{R}^n) \Rightarrow D^\alpha f = \nabla^\alpha f \mathscr{L}^n$.

Since $W^{\alpha,1}(\mathbb{R}^n)$ is closed w.r.t. pointwise convergence, $S^{\alpha,1}(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$.

Remarkably, if $0 < \beta < \alpha < 1$ then $BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$.

Sets of finite fractional Caccioppoli α-perimeter

In perfect analogy with the standard BV setting, we give the following definition:

Let $\alpha \in (0,1)$ and $E \subset \mathbb{R}^n$ be a measurable set. For any open set $\Omega \subset \mathbb{R}^n$, we let

$$|D^\alpha \chi_E|(\Omega) = \sup\left\{ \int_E \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\}$$

be the fractional Caccioppoli α-perimeter of E in Ω. If $|D^\alpha \chi_E|(\Omega) < +\infty$, then E has finite fractional Caccioppoli α-perimeter in Ω.

Note that $E \subset \mathbb{R}^n$ has finite fractional Caccioppoli α-perimeter in Ω if and only if $D^\alpha \chi_E \in \mathcal{M}(\Omega;\mathbb{R}^n)$ and

$$\int_E \operatorname{div}^\alpha \varphi \, dx = -\int_\Omega \varphi \cdot dD^\alpha \chi_E$$

for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$. Note also that $|D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E;\Omega)$.

Another story: we can define a fractional version of De Giorgi's reduced boundary and blow-ups exist!

Asymptotics as $\alpha \to 1^-$

Theorem (Maz'ya-Shaposhnikova & Davila, 2002)

Let $p \in [1, +\infty)$. If $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} (1-\alpha)[f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = c_{n,p} \|\nabla f\|^p_{L^p(\mathbb{R}^n;\, \mathbb{R}^n)}.$$

If $f \in BV(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} (1-\alpha)[f]_{W^{\alpha,1}(\mathbb{R}^n)} = c_{n,1} |Df|(\mathbb{R}^n).$$

Now it is important to observe that

$$\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \sim \frac{1-\alpha}{\omega_n} \quad \text{as } \alpha \to 1^-.$$

Theorem (Comi-S., 2019)

Let $p \in [1, +\infty)$. If $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} \|\nabla^\alpha f - \nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0.$$

If $f \in BV(\mathbb{R}^n)$, then $D^\alpha f \rightharpoonup Df$ and $|D^\alpha f| \rightharpoonup |Df|$ as $\alpha \to 1^-$ and moreover

$$\lim_{\alpha \to 1^-} |D^\alpha f|(\mathbb{R}^n) = |Df|(\mathbb{R}^n).$$

Γ-convergence as $\alpha \to 1^-$

Theorem (Ambrosio-De Philippis-Martinazzi, 2011)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$\Gamma(L^1_{\text{loc}})\text{-}\lim_{\alpha \to 1^-} (1-\alpha) P_\alpha(E;\Omega) = 2\omega_{n-1} P(E;\Omega)$$

for every measurable set $E \subset \mathbb{R}^n$.

Theorem (Comi-S., 2019)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$\Gamma(L^1_{\text{loc}})\text{-}\lim_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) = P(E;\Omega)$$

for every measurable set $E \subset \mathbb{R}^n$.

Theorem (Comi-S., 2019)

Let $\Omega \subset \mathbb{R}^n$ be an open set such that Ω is bounded with Lipschitz boundary or $\Omega = \mathbb{R}^n$. Then

$$\Gamma(L^1)\text{-}\lim_{\alpha \to 1^-} |D^\alpha f|(\Omega) = |Df|(\Omega)$$

for every $f \in BV(\mathbb{R}^n)$.

The space $BV^0(\mathbb{R}^n)$ and the Hardy space

Imitating what we did before, we define

$$BV^0(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |D^0 f|(\mathbb{R}^n) < +\infty\},$$

where

$$|D^0 f|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \,\mathrm{div}^0\varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\}.$$

Here div^0 is the dual operator of $\nabla^0 = I_1\nabla = R$, the Riesz transform

$$Rf(x) = \mu_{n,0} \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(y - x)}{|y - x|^{n+1}} \, dy, \quad x \in \mathbb{R}^n,$$

(in the principal value sense).

We can prove that $BV^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, where

$$H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : Rf \in L^1(\mathbb{R}^n;\mathbb{R}^n)\}$$

is the (real) Hardy space, with $D^0 f = Rf\,\mathscr{L}^n$ as measures for $f \in BV^0(\mathbb{R}^n)$.

Asymptotics as $\alpha \to 0^+$

Theorem (Maz'ya-Shaposhnikova, 2002)

Let $p \in [1, +\infty)$. If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \alpha \, [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = c_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$

Now it is important to observe that $\mu_{n,\alpha} \to \mu_{n,0}$ as $\alpha \to 0^+$ (no rescaling!).

Theorem (Bruè-Calzi-Comi-S., in preparation)

Let $p \in (1, +\infty)$. If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \|\nabla^\alpha f - Rf\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)} = 0.$$

If $f \in H^1(\mathbb{R}^n) \cap \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then (actually, with H^1 norm)

$$\lim_{\alpha \to 0^+} \|\nabla^\alpha f - Rf\|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} = 0.$$

If $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \alpha \int_{\mathbb{R}^n} |\nabla^\alpha f(x)| \, dx = n\omega_n \mu_{n,0} \left| \int_{\mathbb{R}^n} f(x) \, dx \right|.$$

Fractional interpolation inequalities

We prove $\nabla^\alpha \to R$ strongly as $\alpha \to 0^+$ via fractional interpolation inequalities.

Theorem (Bruè-Calzi-Comi-S., in preparation)

Let $\alpha \in (0,1]$. There exists $c_{n,\alpha} > 0$ such that

$$|D^\beta f|(\mathbb{R}^n) \le c_{n,\alpha} \|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} |D^\alpha f|(\mathbb{R}^n)^{\frac{\beta}{\alpha}}$$

for all $\beta \in [0,\alpha]$ and all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$.

Theorem (Bruè-Calzi-Comi-S., in preparation)

Let $p \in (1,+\infty)$. There exists $c_{n,p} > 0$ such that

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,p} \|\nabla^\gamma f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$. If $\gamma = 0$, then

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,p} \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta}{\alpha}}$$

for all $0 \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$.

Open problems and research directions

About sets and perimeter.

- Is there $\chi_E \in BV^\alpha(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n)$?
- Can one characterise blow-ups? Are they unique?
- Are balls isoperimetric sets?
- What about minimal surfaces?

About interpolation.

- What is the real interpolation space between $BV(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$? Note that
$$(H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{p,\alpha} \subset (L^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{p,\alpha} = B^\alpha_{1,p}(\mathbb{R}^n) \supset W^{\alpha,1}(\mathbb{R}^n)$$
and that $B^\alpha_{1,1}(\mathbb{R}^n) = W^{\alpha,1}(\mathbb{R}^n)$.
- Is $BV^\alpha(\mathbb{R}^n)$ the complex interpolation space between $BV(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$?

About the general theory. What is the "right" definition of BV^α on an open set?

Thank you for your attention!