A distributional approach to fractional Sobolev spaces and fractional variation

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No paradoxes without utility

Around 1650 Newton and Leibniz discovered Calculus and nowadays derivative is a basic tool of any mathematician.

Somewhat surprisingly, the first appearance of the concept of a fractional derivative is found in a letter written to de l'Hôpital by Leibniz in 1695!

What is the "half derivative" of x? It's $\dfrac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = c\sqrt{x}$ ($c = \frac{2}{\sqrt{\pi}}$ Lacroix, 1819).

Leibniz's answer to De L'Hôpital, 30 September 1695:

"Il y a de l'apparence qu'on tirera un jour des consequences bien utiles de ces paradoxes, car il n'y a gueres de paradoxes sans utilité."

"This is an apparent paradox from which, one day, useful consequences will be drawn, since there are no paradoxes without utility."

Three famous examples

Let us recall the three most famous fractional derivatives:

Lacroix (1819): $$\frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Riemann-Liouville (1832-1847): $${}^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha} \, d\tau$$

Caputo (1967): $${}^{C}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha} \, d\tau.$$

Some observations:

- they are defined just for functions of one variable;
- only Caputo's derivative kills constants;
- Caputo's derivative requires f to be differentiable!

Question: What about fractional gradient? Can we take $\left(D^{\alpha,1}, \dots, D^{\alpha,n}\right)$?

Be careful: "coordinate approach" gives operator not invariant by rotations!

Šilhavý's approach: invariance properties

Recently, Šilhavý proposed that a "good" fractional operator should satisfy:

- invariance with respect to translations and rotations;
- α-homogeneity for some $\alpha \in (0,1)$;
- mild continuity on suitable test space, e.g. C_c^∞ or Schwartz's space $\mathscr{S}$.

Idea behind: fractional operators should have a physical meaning!

For $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we consider

$$\nabla^\alpha f(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(y-x)}{|y-x|^{n+\alpha+1}}\, dy,$$

and

$$\operatorname{div}^\alpha \varphi(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\, dy.$$

Theorem (Šilhavý, 2018)

Up to mult. const., ∇^α and $\operatorname{div}^\alpha$ are determined by these three properties.

The operators ∇^α and div^α are dual, in the sense that

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

The operators ∇^α and div^α satisfy $-\mathrm{div}^\alpha \nabla^\beta = (-\Delta)^{\frac{\alpha+\beta}{2}}$.

If we let

$$I_\alpha u(x) := \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}} \, dy$$

be the fractional Riesz potential of $u \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m)$, then

$$\nabla^\alpha f = I_{1-\alpha} \nabla f, \qquad \mathrm{div}^\alpha \varphi = I_{1-\alpha}\,\mathrm{div} \varphi.$$

Hence $\nabla^\alpha f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\mathrm{div}^\alpha \varphi \in L^1(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

Relax: ∇^α and div^α are well posed also for Lip_c-regular test functions.

A little bit of literature

The operator $\nabla^\alpha \equiv \nabla I_{1-\alpha}$ has in fact a long story:

- Horváth, 1959 (earliest reference up to knowledge);
- implicitly mentioned in Nikol'ski-Sobolev, 1961;
- non-local continuum mechanics by Edelen-Green-Laws, 1971;
- non-local porous medium equation Caffarelli-Soria-Vazquez, 2011-13, and Biler-Imbert-Karch, 2015;
- fractional PDE theory and "geometric" inequalities by Shieh-Spector, Ponce-Spector, Schikorra-Spector-Van Schaftingen all after 2015;
- distributional approach by Šilhavý, 2018.

Leibniz's rules for ∇^α and $\operatorname{div}^\alpha$

For any $f, g \in C_c^\infty(\mathbb{R}^n)$, we have it holds

$$\nabla^\alpha(fg) = f\nabla^\alpha g + g\nabla^\alpha f + \nabla^\alpha_{\mathrm{NL}}(f,g),$$

where

$$\nabla^\alpha_{\mathrm{NL}}(f,g)(x) := \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(g(y)-g(x))(y-x)}{|y-x|^{n+\alpha+1}}\,dy.$$

For any $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we also have it holds

$$\operatorname{div}^\alpha(f\varphi) = f\operatorname{div}^\alpha\varphi + \varphi\cdot\nabla^\alpha f + \operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi),$$

where

$$\operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi)(x) := \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy.$$

The operators $\nabla^\alpha_{\mathrm{NL}}$ and $\operatorname{div}^\alpha_{\mathrm{NL}}$ have a strong non-local behaviour.

The operators ∇^α and $\operatorname{div}^\alpha$ commute with convolutions. Leibniz's rules allow for cut-off approximation arguments (careful on non-local terms!).

For $p \in [1, +\infty)$ and $\alpha \in (0,1)$, we let

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p\alpha}} \, dx \, dy < +\infty \right\}.$$

A measurable set $E \subset \mathbb{R}^n$ has finite fractional perimeter if

$$P_\alpha(E) = [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)} = 2 \int_{\mathbb{R}^n \setminus E} \int_E \frac{1}{|x-y|^{n+\alpha}} \, dx \, dy < +\infty$$

and we define its fractional perimeter in an open set $\Omega \subset \mathbb{R}^n$ as

$$P_\alpha(E;\Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}} \, dxdy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}} \, dxdy.$$

Notice that we have the extension $\nabla^\alpha : W^{\alpha,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n; \mathbb{R}^n)$, since

$$\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \mu_{n,\alpha} [f]_{W^{\alpha,1}(\mathbb{R}^n)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

Analogously, we have the extension $\operatorname{div}^\alpha : W^{\alpha,1}(\mathbb{R}^n; \mathbb{R}^n) \to L^1(\mathbb{R}^n)$.

Fractional variation and the space $BV^\alpha(\mathbb{R}^n)$

We define

$$BV^\alpha(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty\},$$

where

$$|D^\alpha f|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\}.$$

In perfect analogy with the classical BV framework:

- $BV^\alpha(\mathbb{R}^n)$ is a Banach space and its norm is l.s.c. w.r.t. L^1-conv.;
- $C^\infty(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ are dense subspaces of $BV^\alpha(\mathbb{R}^n)$;
- given $f \in L^1(\mathbb{R}^n)$, $f \in BV^\alpha(\mathbb{R}^n) \iff \exists D^\alpha f \in \mathcal{M}(\mathbb{R}^n;\mathbb{R}^n)$ s.t.

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f \quad \text{for any } \varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n);$$

- unif. bounded seq. in $BV^\alpha(\mathbb{R}^n)$ admit limits in $L^1(\mathbb{R}^n)$ w.r.t. L^1_{loc}-conv.;
- for $n \ge 2$ we have $BV^\alpha(\mathbb{R}^n) \subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ (GNS inequality).

$BV(\mathbb{R}^n) \subset W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$

We know $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ and $f \in W^{\alpha,1}(\mathbb{R}^n) \Rightarrow D^\alpha f = \nabla^\alpha f \mathscr{L}^n$.

We define the distributional fractional Sobolev space

$$S^{\alpha,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : \exists\, \nabla^\alpha_w f \in L^p(\mathbb{R}^n;\mathbb{R}^n)\}.$$

Given $f \in BV^\alpha(\mathbb{R}^n)$, we have $f \in S^{\alpha,1}(\mathbb{R}^n) \iff |D^\alpha f| \ll \mathscr{L}^n$, so

$$D^\alpha f = \nabla^\alpha_w f\, \mathscr{L}^n.$$

We are able to prove that $BV^\alpha(\mathbb{R}^n) \setminus S^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$.

$W^{\alpha,1}(\mathbb{R}^n)$ is closed w.r.t. pointwise conv. $\Rightarrow S^{\alpha,1}(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$.

If $f \in BV(\mathbb{R}^n)$ then $f \in BV^\alpha(\mathbb{R}^n)$, with

$$\nabla^\alpha f(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{dDf(y)}{|y-x|^{n+\alpha-1}} = I_{1-\alpha} Df(x)$$

for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Remarkably, if $0 < \beta < \alpha < 1$ then $BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$.

In analogy with the standard BV setting, we give the following definition:

Let $\alpha \in (0,1)$ and $E \subset \mathbb{R}^n$. For any open set $\Omega \subset \mathbb{R}^n$, we let

$$|D^\alpha \chi_E|(\Omega) = \sup\left\{ \int_E \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\}$$

be the fractional Caccioppoli α-perimeter of E in Ω. If $|D^\alpha \chi_E|(\Omega) < +\infty$, then E has finite fractional Caccioppoli α-perimeter in Ω.

Note that $E \subset \mathbb{R}^n$ has finite fractional Caccioppoli α-perimeter in Ω if and only if $D^\alpha \chi_E \in \mathcal{M}(\Omega;\mathbb{R}^n)$ and

$$\int_E \operatorname{div}^\alpha \varphi \, dx = -\int_\Omega \varphi \cdot dD^\alpha \chi_E$$

for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$.

Fractional reduced boundary

It is now natural to give the following definition:

Let $E \subset \mathbb{R}^n$ be a set with finite fractional Caccioppoli α-perimeter in Ω. A point $x \in \Omega$ belongs to the fractional reduced boundary of E (inside Ω) if

$$x \in \operatorname{supp}(D^\alpha \chi_E) \qquad \text{and} \qquad \exists \lim_{r\to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} \in \mathbb{S}^{n-1}.$$

We thus let $\mathscr{F}^\alpha E$ be the fractional reduced boundary of E and define

$$\nu_E^\alpha \colon \Omega \cap \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}, \quad \nu_E^\alpha(x) := \lim_{r\to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))}, \quad x \in \Omega \cap \mathscr{F}^\alpha E,$$

the inner unit fractional normal to E (inside Ω).

We thus have the following Gauss-Green formula

$$\int_E \operatorname{div}^\alpha \varphi \, dx = -\int_{\Omega \cap \mathscr{F}^\alpha E} \varphi \cdot \nu_E^\alpha \, d|D^\alpha \chi_E|.$$

for all $\varphi \in \operatorname{Lip}_c(\Omega; \mathbb{R}^n)$.

If $E \subset \mathbb{R}^n$ satisfies $P_\alpha(E;\Omega) < +\infty$, then

$$|D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E;\Omega)$$

and $D^\alpha \chi_E = \nu_E^\alpha \, |D^\alpha \chi_E| = \nabla^\alpha \chi_E \, \mathscr{L}^n$. Moreover, if $\chi_E \in BV(\mathbb{R}^n)$, then

$$\nabla^\alpha \chi_E(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{\nu_E(y)}{|y-x|^{n+\alpha-1}} \, d|D\chi_E|(y)$$

for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Be careful! We have

$$P_\alpha(E;\Omega) < +\infty \Rightarrow \mathscr{L}^n(\Omega \cap \mathscr{F}^\alpha E) > 0$$

including even the case $\chi_E \in BV(\mathbb{R}^n)$. In other words, the non-local operator ∇^α produces a diffuse fractional boundary in the $W^{\alpha,1}$ regime.

Example: $E = (a,b) \subset \mathbb{R} \Rightarrow \mathscr{F}^\alpha E = \mathbb{R} \setminus \left\{\frac{a+b}{2}\right\}$!

Existence of blow-ups

Let $\mathrm{Tan}(E, x)$ be the set of all tangent sets of E at x, i.e. the set of all limit points in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-topology of the family

$$\left\{\frac{E-x}{r} : r > 0\right\} \quad \text{as } r \to 0.$$

Theorem (Comi-S., 2018)

Assume E has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$.

- $\mathrm{Tan}(E, x) \neq \varnothing$ for all $x \in \mathscr{F}^\alpha E$.
- $F \in \mathrm{Tan}(E, x) \Rightarrow F$ has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ and $\nu_F^\alpha(y) = \nu_E^\alpha(x)$ for $|D^\alpha \chi_F|$-a.e. $y \in \mathscr{F}^\alpha F$.

Asymptotics as $\alpha \to 1^-$

Now it is important to observe that

$$\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \sim \frac{1-\alpha}{\omega_n} \quad \text{as } \alpha \to 1^-.$$

Proposition (Comi-S., in preparation)

Let $p \in [1, +\infty]$. If $f \in C_c^2(\mathbb{R}^n)$ and $\varphi \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$, then

$$\lim_{\alpha \to 1^-} \|\nabla^\alpha f - \nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0, \qquad \lim_{\alpha \to 1^-} \|\operatorname{div}^\alpha \varphi - \operatorname{div} \varphi\|_{L^p(\mathbb{R}^n)} = 0.$$

Theorem (Comi-S., in preparation)

If $f \in BV(\mathbb{R}^n)$, then $D^\alpha f \rightharpoonup Df$ and $|D^\alpha f| \rightharpoonup |Df|$ as $\alpha \to 1^-$ and moreover

$$\lim_{\alpha \to 1^-} |D^\alpha f|(\mathbb{R}^n) = |Df|(\mathbb{R}^n).$$

Γ-convergence to Euclidean perimeter

Theorem (Comi-S., in preparation)

Let $\Omega \subset \mathbb{R}^n$ be an open set. If $\chi_{E_\alpha} \to \chi_E$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $\alpha \to 1^-$, then

$$P(E;\Omega) \le \liminf_{\alpha \to 1^-} |D^\alpha \chi_{E_\alpha}|(\Omega).$$

Theorem (Comi-S., in preparation)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. If $P(E) < +\infty$ and $|D\chi_E|(\partial\Omega) = 0$, then

$$\limsup_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) \le P(E;\Omega).$$

Corollary (Comi-S., in preparation)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. For every measurable set $E \subset \mathbb{R}^n$, we have

$$\Gamma(L^1_{\text{loc}})\text{-}\lim_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) = P(E;\Omega).$$

Open problems

This new distributional approach aims to deal with a large variety of problems:

- better characterisation of blow-ups (uniqueness?);
- Structure Theorem for (a subset of) $\mathscr{F}^\alpha E$ in the spirit of De Giorgi's Theorem;
- Gauss-Green & integration-by-part formulas for sets of (locally) finite fractional Caccioppoli α-perimeter;
- link between $BV^\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $W^{\alpha,1}(\mathbb{R}^n)$?
- what about minimal surfaces? fractional calibration? any regularity?
- isoperimetric sets (balls? symmetrisation?);
- asymptotics for $\beta \to \alpha^-$ given any $\alpha \in (0,1)$;
- asymptotics $\alpha \to 0^+$ (Hardy space?);
- good definition of BV^α functions on a general open set.

Thank you for your attention!